Symmetry and equivalence

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As other contributions to this volume also testify, the notions of symmetry and equivalence are closely connected. This paper is devoted to exploring this connection and its relevance to the symmetry issue, starting from its historical roots. In fact, it emerges as an essential and constant feature in the evolution of the modern notion of symmetry: at the beginning, as a specific relation between symmetry and equality; in the end, as a general link between the notions of symmetry, equivalence class, and transformation group.

1 Symmetry and equality

Weyl's 1952 classic text on symmetry starts with the following distinction between two common notions of symmetry:

> If I am not mistaken the word *symmetry* is used in our everyday language in two meanings. In the one sense symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. ... The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: *bilateral symmetry*, the symmetry of left and right ...

Bilateral symmetry is in fact a particular case of the scientific notion of symmetry, the symmetry being defined as invariance with respect to a transformation group (in the case of bilateral symmetry, the group of spatial reflections). This is the symmetry we are interested in here: the notion which, thanks to its group-theoretic characterization, has proved so successful in twentieth-century physics. But how did it develop? We know that the symmetry of the Greeks and Romans, current until the end of the Renaissance, was the one grounded on proportion relations. But we also know, thanks to a 1673 text by the French architect and physician Claude Perrault (the brother of the more famous Charles), that the second notion of symmetry (not yet in its group-theoretic formulation, of course) became commonly used among his contemporaries. In fact, in presenting his French translation of Vitruvius's *De Architectura*, Perrault introduced a very similar distinction to the one stated above, i.e. the distinction between an *ancient notion* of symmetry, based on proportions (and used for example by Vitruvius), and a *modern notion*, defined as a relation of equality between parts that are opposed – the most familar case being the relation of equality between the right and left parts of a figure (Perrault, 1673, p. 10).

We thus know that symmetry, in its initial modern sense, was closely related to spatial equality. What exactly is the nature of this relationship? In particular, how does it change in the conceptual development of symmetry from its initial meaning (a relation of equality between opposed parts) to its group-theoretic definition? This section and the next are devoted to exploring this issue.

The equality of the parts is indeed the distinguishing feature of the modern notion in its original version: it marks the difference with respect to the ancient notion (the previous role of symmetry being that of "according" parts of different size and form by means of proportions). At the same time, the equality of the parts is what made possible the mathematical development of the symmetry notion. Parts that are equal can be exchanged or substituted: it thus became possible to introduce mathematical operations (such as translations, rotations, and reflections) for describing how, in fact, the parts could be exchanged.

The equality of the parts is, of course, not enough. Symmetry is something more than simple repetition. It is characterized by the sort of regularity inherent in the arrangement of equal parts. In its modern sense, symmetry is always a property of the whole: in the case of a spatial arrangement of equal parts, a property of the whole arrangement. The relationship between symmetry and equality is therefore to be understood with respect to the whole: the parts of a symmetric arrangement are each equal to one another and with respect to the whole; that is, the entire configuration does not change when the parts are exchanged. The symmetric character of an arrangement of equal parts has thus to do with the fact that the whole figure does not change when the parts are exchanged by means of some operations. These operations are then identified as the "symmetry operations" of the figure, and the type of the symmetry (translational, rotational, etc.) is given by the type of symmetry operations considered (translations, rotations, etc.).

Symmetry could thus be characterized in terms of the invariance of the figure studied under specified operations. The next step, in its conceptual history, was of a purely mathematical nature: the nineteenth-century introduction of the algebraic concept of a *group* and the subsequent development of the theory of transformation groups. The symmetry operations ("symmetry transformations") of a figure were found to satisfy the conditions for constituting a transformation group.[1] This led to the final stage in the evolution of the modern notion of symmetry: its general mathematical definition as "invariance under a group of transformations".

2 Symmetry, equivalence, and group

What about the initial equality of parts? We started with the familiar case of a spatial arrangement of equal parts, and arrived at the abstract group-theoretic notion of symmetry applied in modern science. But the substance has not changed: the "equality of parts" is still there, but in the form of an *equivalence relation*, i.e. the equivalence relation between the elements that are connected to each other by the symmetry transformations of the "whole".

To see better what this means, let us stay on familiar ground by taking again the case of a symmetric arrangement of parts that are related to each other by an equality relation. For a relation of this nature we expect that it is

1) *reflexive*: each part is equal to itself;

2) *symmetric*: if part A is equal to part B, then B is equal to A;

3) *transitive*: if part A is equal to part B and part B is equal to part C, then A is equal to C.

Equality is in fact the most obvious example of an *equivalence relation* (i.e. a relation that is reflexive, symmetric and transitive). Now, this has the following interesting implications for the symmetry operations of the whole figure (the operations exchanging the equal parts in such a way that the whole remains invariant):

[1] A group is defined to be a set G with a product $\bullet$ which assigns to each pair (g_1, g_2) of elements in G one element $g_1 \bullet g_2$ in G, and is associative, has a neutral element, and for which each element has an inverse.

1) the existence of the identity operation, corresponding to the reflexive property of the relation between the parts;

2) the existence of the inverse operation, corresponding to the symmetric property of the relation between the parts;

3) the closure of the product between any two operations, corresponding to the transitive property of the relation between the parts.

In substance, the fact that the parts are related by means of an equivalence relation (which is at the same time *the* equivalence relation between the parts, ensuring their interchangeability in the considered context) corresponds to the fact that the family of operations transforming the parts into each other while leaving the whole invariant satisfies the conditions for constituting a group (i.e. the existence of the identity and inverse operations, and the closure of the product).[2] In other words, the two main features for capturing the essence of a symmetry – the invariance of the whole and the equivalence of the parts (at the same time between each other and with respect to the whole) – are mathematically mirrored in the group properties. It is worth underlining that not any equivalence relation will do, but only *the* equivalence relation which is such as to ensure the invariance of the whole when the equivalent parts are exchanged. This is the difference between a symmetry and a bare repetition. There are conditions regulating the way in which the parts are equivalent: namely, the conditions dictated by the invariance of the whole under the transformations relating the equivalent elements, mathematically translated into the group conditions for the symmetry transformations.

The link thus obtained between the group nature of the symmetry operations and the fact that they exchange elements that are equivalent is in fact valid in general, independently of whether we are considering the symmetry properties of familiar spatial figures or of abstract mathematical relations. What the equivalent elements are, how they are related and in which way the symmetry of the whole is obtained of course depend on the specific situation considered. But the connection that can be established between the notions of symmetry, equivalence, and group is a general feature.

More can be said, in fact. We know that, given an equivalence relation on a set A, the set of all the elements equivalent to a given element x of

[2]For a more detailed discussion see for example Yaglom (1988), pp. 114-16.

A forms the *equivalence class* of x. A lemma says that if two elements are equivalent, they have the same equivalence class. Then we have a theorem stating that, if we have an equivalence relation on A, the family of all the equivalence classes is a *partition* of A.[3] Proceeding the other way round, we may begin with a partition and end up with an equivalence relation. It is thus legitimate to conclude that the two concepts of *equivalence relation* and *partition* are "twin aspects of the same structure on sets" (Pinter, 1982, p. 120). Now, as we have seen, the elements exchanged with one another by the symmetry transformations of a figure (or whatever the "whole" considered is) are connected by an equivalence relation. They thus form an equivalence class. More generally, we can then say that the presence of a symmetry group in a given context is related to a partition into equivalence classes.[4] The symmetry transformations, by permuting the elements in the same equivalence class, leave this structure invariant. At the same time, they leave those features invariant that are common to all the elements constituting the same equivalence class. Note that these features are exactly the ones necessary to characterize the class as such within the considered context. Two fundamental aspects of the notion of symmetry clearly emerge here: a) its generality: what remain invariant under symmetry transformations are properties of classes, not of individual elements;[5] and b) its "structuring" role: symmetry at the same time defines and preserves structures.[6]

3 Equivalence and irrelevance

A symmetry, we have seen, corresponds to a situation of equivalence with respect to a given context: elements that are connected with one another by symmetry transformations form an equivalence class. How does this apply to physics? A physical symmetry corresponds to the equivalence of a number of elements with respect to the physical theory considered. The

[3] For details see for example Pinter (1982), chapter 12.

[4] For a slightly different way of presenting the connection between the concepts of equivalence relation, partition, and group transformations, see van Fraassen (1989), chapter 10, section 3, who arrives at the conclusion that these three concepts "amount really to the same concept" (p. 246).

[5] Note that this is the basis for the possibility of classification on the grounds of symmetry properties: one of the most important functions of symmetry in science. On this point see the introduction to this volume.

[6] Whence the use, in some of the literature, to define symmetries as transformations that preserve structure. See, in particular, Ismael and van Fraassen, this volume.

contributions to this volume offer a good variety of examples: permutation symmetry and the equivalence of so-called identical particles (French and Rickles, Huggett, Saunders), the equivalence of spacetime points connected with relativity principles (Norton), and the equivalence of phase space points lying in the same gauge orbit (Redhead, Earman), to mention some. Note that the role of the "whole" is now played by the physical theory; more precisely, by its dynamical equations. The theory gives the same description – that is, the fundamental dynamical equations do not change – when the equivalent elements are exchanged with one another by the transformations of the symmetry group. This is the received view in the physics literature.

Different questions naturally arise at this point. This paper focuses on the following one: how is the equivalence of the elements to be understood? Generally speaking, it is quite common to think that if some elements are equivalent from the viewpoint we are considering, we do not need to take all of them into account. One of the equivalent elements can be representative of the others. In which way representative and with respect to what needs to be clarified, of course. But the moral is general: in some way, equivalence seems to go with *irrelevance* or *redundancy*, the presence of equivalent elements suggesting the presence of irrelevant or redundant features in the context considered.

A concrete example of this moral is the standard attitude among physicists towards the symmetries postulated by special relativity, that is the global symmetries of the spacetime continuum. Take, for example, the case of (spatial) translations and the corresponding symmetry (homogeneity of space): positions are equivalent to one another from the viewpoint of the physical description, the dynamical equations not varying in form under translations. Now, this is usually taken to indicate the irrelevance of an absolute position to the physical description: all space points are equivalent as far as the physics (i.e., the dynamical equations) is concerned. One of them is representative of the whole class, and the symmetry transformations are the mathematical tools by means of which this is made possible; or, in other words, a reference frame located at any position is representative of all the others, the translational symmetry ensuring the invariance of the physics when passing to another reference frame by means of symmetry transformations. The same is commonly argued in the case of the other global spacetime symmetries. Let us just mention what Eugene Wigner writes in this respect: "the older laws of invariance", the ones "which found their perfect, and perhaps final, formulation in the special theory of relativity ... postulate, in addition to the irrelevance of the absolute position and time of an event, the irrelevance of its orientation and finally, the irrelevance of its state of motion, as long as this remains uniform, free of rotation, and on a straight line" (Wigner, 1967, pp. 4-5).

As already said, in what way the equivalent elements are connected to the presence of irrelevant features in the theory depends on the specific case studied. Moreover, the nature of these "irrelevancies" is not the same for all types of symmetries, the major difference being between global and local (i.e. dependent on spacetime points) symmetries. In the global case, all the equivalent elements (for example, all the equivalent space locations) have the same physical status ("reality"). But this is controversial in the local case, where, as we shall see in the next section, attributing the same reality to the equivalent elements leads to indeterminism.

It is, however, a fact that, independently of how it is realized, the connection between symmetry, equivalence, and irrelevance is indeed a close one. On this basis, a natural position on the symmetry issue is the one relating the presence of physical symmetries to the presence of irrelevant elements in the physical description. This has an immediate empirical counterpart in terms of nonobservability. The idea is that irrelevant theoretical features make no observable difference. Symmetries are thus connected with the presence of nonobservable quantities in the physical description, with the corollary that the empirical violation of a symmetry (the paradigmatic case is parity violation in the case of weak interactions) is intended in the sense that "what was thought to be a non observable turns out to be actually an observable" (Lee, 1988, p. 178).

The view that "the root of all symmetry principles lies in the assumption that it is impossible to observe certain basic quantities" (*ibid.*) is quite popular, especially among physicists. It leaves, however, many questions open, the most important of which concerns the nature and origin of the irrelevant/nonobservable features.

P. A. M. Dirac, in his 1930 preface to his masterpiece *The principles of quantum mechanics*, gives us a possible clue (p. vii):

> [Nature's] fundamental laws control a substratum of which we cannot form a mental picture without introducing irrelevancies. The formulation of these laws requires the use of the mathematics of transformations.

This may be taken as an illustrious example of the following view: in describing the physical world, because of our inherent epistemic limits in relation to the nature of what we are trying to describe, we introduce irrelevant theoretical elements and this is signalled by the presence of symmetries. This is how far the connection between symmetry and equivalence has brought us until now. But this is still very vague: in what way are these irrelevancies introduced? And what is peculiar to the ones signalled by symmetries? Two ways of addressing these questions are examined separately and then brought together in the concluding section.

4 Symmetry and arbitrariness

Two main points have emerged so far:

- a symmetry gives rise to equivalence classes (the elements that are connected with one another by symmetry transformations form an equivalence class);
- symmetries are related in some way to the presence of irrelevant features in the physical description.

Now, there are two sorts of arbitrariness involved here.

- On the one side, there is a freedom associated with the fact that a symmetry gives rise to equivalence classes: the freedom to choose one element as representative of the class; which one is completely arbitrary.
- On the other side, there is a certain arbitrariness in the distinction between what is relevant and what is irrelevant with respect to the physical situation studied. What is relevant or irrelevant may also depend on the level of detail, or the "scale" (spatial, temporal, etc.), at which the situation is considered.

Let us explore both cases in some more detail, to see how far this takes us.

a) *Gauge freedom and constraints*
The link between symmetry and equivalence is general but, as some of the contributions to this volume clearly show (see especially Martin), there are subtle interpretative questions related to the difference between global and local (or gauge) symmetries. We focus here on the gauge case, for the simple

reason that it has greater interest for the purpose of this paper. In the local symmetry case, the freedom to choose one element as representative of the equivalence class ("gauge freedom", in the literature) corresponds to the presence of surplus "unphysical" degrees of freedom in the theory, an example of what has been called by Michael Redhead "surplus structure".[7] There is now a quite intensive discussion in the literature about the significance of these surplus degrees of freedom.[8] The main issue is whether to consider them literally (with indeterminism as a consequence),[9] or to eliminate them in some way,[10] or to search for another type of solution.

A convenient framework for examining the nature and role of the arbitrary features in the gauge case is the *theory of constrained Hamiltonian systems*, going back to Dirac's seminal works in the 1950s. There is in fact an important relation between gauges and constraints, as pointed out also in other contributions to this volume (see especially Earman). On the one side, it is shown (by using Noether's second theorem) that all systems with a gauge invariance are necessarily singular systems with constraints.[11] On the other side, as first conjectured by Dirac in the 1950s, all so-called first-class constraints are demonstrated to be generators of gauge transformations. The presence of arbitrary features in the mathematical framework plays in fact a crucial role in obtaining these results, as will be briefly illustrated here below.

Gauge theories are an example of singular theories, that is theories describing a physical system by more variables than there are physically independent degrees of freedom. In such cases it is easy to show that the general solutions of the Hamiltonian equations of motion with given initial conditions depend on arbirary functions of time. Let us see briefly how Dirac's

[7]The significance of this surplus structure, first considered in Redhead (1975), is thoroughly analysed in Redhead's contribution to this volume.

[8]See for example Earman, Martin, and Redhead, this volume, and references therein.

[9]Wallace, this volume, is devoted to clarifying this point.

[10]For example, by moving from the original phase space to a "reduced space", the points of which are equivalence classes of the original phase space points that are related by symmetry transformations.

[11]According to Noether's second theorem, for a gauge invariant system – that is, a system invariant under transformations defining a simply connected continuous group, whose parameters are r arbitrary functions of time (or spacetime) – there exist r independent identities of the Euler-Lagrange derivatives of the Lagrange function. These identities are consequences of the gauge invariance and define (Lagrangian) constraints on the system. That gauge invariant systems are singular systems may be easily seen by analysing these identities.

analysis proceeds in the simple case of a system with a finite number N of degrees of freedom, with general coordinates q_n $(n = 1, ..., N)$ and velocities $\dot{q}_n = dq_n/dt$.[12]

In the singular case, the velocities $\dot{q}_n$ are not uniquely determined in terms of the coordinates q_n and momenta p_n.[13] This means that not all momenta are independent functions of the velocities; that is, there must exist a set of relations of the form

$$\phi_m(q_n, p_n) = 0, \quad m = 1, ..., M \ ,$$

called by Dirac the "primary constraints" of the Hamiltonian formalism.[14] Now, given a set of N phase space degrees of freedom and a set of M primary constraints, time evolution may be generated not only by the canonical Hamiltonian $H_o = p_n \dot{q}_n - L$, but (since we may add to it any linear combination of the ϕs which are zero) by a generalized Hamiltonian of the form

$$H_* = H_o + u_m(q_n, p_n; t)\phi_m(q_n, p_n) \ ,$$

where ϕ_m are all primary constraints and u_m are arbitrary functions of time and of the phase space variables. Starting with this expression and carrying out his analysis by imposing all the consistency requirements of the theory, Dirac then arrives at the following final expression for the generalized Hamiltonian:

$$H_* = H + v_a(t)\phi_a \ ,$$

[12] Here I closely follow Dirac's 1964 *Lectures on quantum mechanics* (the classic reference in this regard). More details on what follows in the text can be found in E. Castellani, 'Dirac on gauges and constraints', PITT-PHIL-SCI00000573.

[13] In Lagrangian terms, this is the situation expressed by the singularity of the Hessian matrix of the Lagrange function, that is by the fact that

$$det\left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^{n'}}\right) = 0 \ .$$

[14] Dirac distinguishes between *primary* and *secondary* constraints (depending on whether their definition is independent of the Lagrangian equations of motion or not), and *first-class* and *second-class* constraints (depending on whether their Poisson brackets with all other constraints is weakly vanishing or not). The really important distinction for the treatment of constrained systems is the second one. For details see Dirac (1964), pp. 14-18.

where $H = H_o + U_m \phi_m$ and $\phi_a = V_{am} \phi_m$. The U_m and V_{am} are functions of the phase space variables satisfying to given consistency equations, while the $v_a(t)$ are *arbitrary functions of time* (their number being equal to the number of primary first-class constraints, usually less than the number of all constraints).[15]

The result is thus that, although all consistency requirements have been satisfied, the theory still has arbitrary coefficients which may depend on time, the $v_a(t)$. This means that the general solution of the Hamiltonian equations of motion with given initial conditions depends on arbirary functions of time. What follows from this? According to Dirac, the arbitrariness in the choice of the functions $v_a(t)$ implies that the different trajectories in phase space obtained under time evolution for given initial conditions but for different $v_a(t)$ should be considered as representing the same configuration of the system. It was then conjectured by Dirac (and later demonstrated) that different points in phase space representing the same state of the system are related to one another by gauge transformations that are generated by the first-class constraints of the theory. This result – constraints are generators of gauge transformations – is indeed the upshot of this part of Dirac's analysis.

It is thus made quite precise, in Dirac's analysis, in what the arbitrariness peculiar to the above sort of situation consists, and how the connected surplus theoretical features enter into the theory: in particular, many different phase space points and trajectories representing the same time dependent configuration of the system. The Hamiltonian treatment à la Dirac shows how they are related to one another by gauge transformations. Now, how should we deal with the equivalent or redundant descriptions we have thus obtained? In agreement with Dirac's position, the common answer in the physical literature is: by keeping the physically meaningful degrees of freedom only (ideally, by means of a description of the dynamical evolution of the system in terms of the reduced space representing the truly distinct possible configurations of the system). In gauge terms, this corresponds to the common strategy of attributing physical meaning to gauge invariant quantities only. The underlying idea is that in a theory where the system is described by more variables than the number of independent degrees of freedom, "the physically meaningful degrees of freedom reemerge as being those invariant under a transformation connecting the variables (gauge transformation)"(Henneaux and Teitelboim, 1992, p. 1). The philosophy, in substance,

[15] For details see Dirac (1964), pp. 13-16.

is the following: for given reasons we introduce extra variables (surplus theoretical features or "irrelevancies", in Dirac's words) in the theory, and at the same time we bring in a (gauge) symmetry "to extract the physically relevant context" (*ibid.*).[16]

This is, at first sight, a very strong epistemic attitude towards the meaning of gauge symmetries. I want to suggest a way of justifying a less strongly epistemic approach which nevertheless respects the basic motivations for the above view. The idea is a sort of compromise: symmetries do enter in our way of describing the physical world, but in a way that is significantly "constrained" by the reality we want to describe. In this regard, particular relevance is to be attributed to the role played by physical scales and related boundaries conditions. This is what the concluding part of this paper tries to suggest, by examining the second sort of arbitrariness related to physical symmetries: namely, the degree of arbitrariness in the distinction between what is relevant and what is irrelevant with respect to a given context.

b) *Symmetries and scales*

As we have said, what is relevant or irrelevant with respect to a given context depends not only on the characteristics of the chosen domain (which is obvious), but also on the level of detail at which this domain is considered. Today's physics offers in fact an important suggestion in this sense; that is, the idea at the basis of a recent view of current quantum field theories as *effective theories*: physics can change as one changes the scale considered, at very different ranges of energy scales we can have remarkably different physics. In recent years, in fact, developments in quantum field theory (QFT) and in the application of renormalization group theory techniques have brought a "change of attitude" in particle physics (Weinberg, 1997, p. 41), which is at the basis of the so-called modern view of QFT: that is, the view that "the most appropriate description of particle interactions in the language of QFT depends on the energy at which the interactions are studied" (Georgi, 1989, p. 446). On this view, current QFTs are understood as *effective field theories*, each effective field theory explicitly referring only to those particles that are actually of importance at the range of energies considered. The key point is the "decoupling" of the physics at the chosen energy scale from the physics at much higher energies. In the QFT framework, this decoupling of

[16]This point of view has been seen by some as problematic in the case of the Aharonov-Bohm effect, where the gauge invariant quantities are the Yang-Mills fields plus the holonomies. For philosophical reflections on this point see Nounou, this volume.

physical phenomena as well as the changing of the effective physical description as the scale changes occur according to specific and precise rules. This is due essentially to some important peculiarities of the local quantum field description and, in particular, to the concept of the *renormalization group* and its deep impact on particle physics.

The effective field theory approach suggests that scale considerations might play an important role also when discussing the meaning of physical symmetries. What is relevant or irrelevant is in fact also related to what physical scale we choose. That is, the choice of physical scale imposes a coarse-graining, which in itself creates equivalence classes (the elements of which are all equivalent to one another with respect to the disregarded or "cut-off" features in the coarse-graining). The cutting-off procedure, by means of which the coarse-graining is effected, thus fixes a distinction between what is being relevant or irrelevant at the chosen scale.

In this way, the arbitrariness implied in the relevant/irrelevant distinction comes to be significantly related to the arbitrariness in the choice of the scale at which the physical situation is studied. The choice of the scale is apparently up to us, so this seems to go well with a purely epistemic attitude towards the meaning of symmetries. But the situation is actually a bit more subtle here. It is true that choosing the scale is up to us, but not *entirely*. The nature of the physical context we want to describe constrains in a significant way our choice. The scale is the one we use in the laboratories, but it is also the one given by the energies typical of the successive states of the universe in its evolution. As is clearly stated in a recent popular book by the physicist Brian Greene (1999, p. 350-51), "the significant differences between the forces as we now observe them were all erased by the extremes of energy and temperature encountered in the very early universe. But as the time went by and the universe expanded and cooled, the formalism of quantum field theory shows that this symmetry would have been sharply reduced ... leading to the comparatively asymmetric form with which we are familiar."

So, although there is an arbitrariness implied in the choice of scale (and hence in the distinction between what is relevant or irrelevant), this arbitrariness is limited by factors external to us, and this is where the realistic turn to the view discussed in this paper appears. It is a fact that what is a good physical description or approximation in a given regime may not be a good description or approximation in another very different regime. In particular, what are "unphysical" surplus features in the appropriate description at a determinate regime may become physically relevant features in a very different regime. The following words by the leading theoretical physicist David Gross (1999, p. 58) offer a good and conclusive illustration of what is meant here: "Today we believe that global symmetries are unnatural ... We now suspect that all fundamental symmetries are local gauge symmetries. Global symmetries are either broken, or approximate, or they are the remnants of spontaneously broken local symmetries."

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