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Sektion 1: Logik und Wissenschaftstheorie

Three-valued logic and modalities: Łukasiewicz's proposal

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The aim of this talk is to show that a (certain) *modal* interpretation of the three-valued system for propositional logic worked out by Łukasiewicz can be provided if and only if we put that system on the philosophical background from which it originally arose, that is, the long debated problem of the logical value of propositions expressing future contingent events.

Whereas the strictly logical significance of the three-valued system sketched in Łukasiewicz [1920] has been widely accepted as the starting point of non-classical (many-valued) systems of propositional logic, the original *motivations* which led the polish philosopher and logician to work out such a system have been often forgotten or neglected by most scholars.

These motivations are exposed and deeply discussed especially in Łukasiewicz [1922] and [1931] and have essentially to do with the following two points: (1) classical two-valued logic leaves *no place* for contingency in the world, that is, the principle of bivalence is the logical counterpart of the ontological thesis of determinism: everything is either necessary or impossible; (2) nevertheless, *we see* in everyday life that there are contingent events, i.e. events which have the possibility of happen as well the possibility of not happen – this second point is also reinforced by the "indeterminist creed" explicitly accepted by Łukasiewicz in [1918].

Now, Łukasiewicz's proposal in order to avoid a deterministic world-view and to admit future contingent events, lies in getting rid of the principle of bivalence and introduce a third logical value, the 'indeterminate', attaching exactly to those propositions which express at present merely possible events (i.e. events which are neither necessary nor impossible), whereas the logical values 'true' and 'false' attach to propositions expressing events whose existence or non existence is (already) decided. The outcome of this choice will be a non-classical logic which is, in Łukasiewicz's view, more *adequate* than the classical (bivalent) one as the basis of a scientific thought concerned with the "real world" and his complexity.

Given these presuppositions it becomes clear that the three-valued system of propositional logic worked out by Łukasiewicz (henceforth: Ł3) comes up as a system of modal logic: a system which takes into account the different modalities which *events* can assume. Further, such a system allows – contrary to what happens in classical systems of propositional logic (henceforth: L2) – a non trivial definition of a *truth-functional* operator of possibility which verifies the classical "modal square" of oppositions as well the most significant modal principles accepted in the logical and philosophical tradition. One has to look in this way at Ł3 as a system of extensional modal logic which opposes to the classical systems of intensional modal logic worked out for example by Lewis-Langford [1932] and von Wright [1951].

Let us briefly put into a sharper focus the way in which Ł3 can be interpreted as a system of modal logic and how such an interpretation has given rise to the difficulties which led most scholars to neglect the "modal content" of such a system.

We can define a propositional monadic operator of possibility 'M' («it is possible that...») as follows:

$$Mp =_{def} \neg p \to p$$

(such a definition was suggested to Łukasiewicz by the young Tarski). Let us indicate the value 'true' by 1, the value 'false' by 0 and the value 'indeterminate' by ½. Given the semantic rules for Ł3 outlined in Łukasiewicz [1920] we have the following three cases corresponding to the possible values of p:

(i) if $p=1$, then $Mp=1$;
(ii) if $p=\frac{1}{2}$, then $Mp=1$;
(iii) if $p=0$, then $Mp=0$.

The first and second case are quite intuitive; the third case looks like problematic and will be discussed in a moment. On the basis of the definition of Mp given above we can also define the operator of necessity 'N' («it is necessary that...») as:

$$Np = \neg M \neg p = \neg(\neg\neg p \to \neg p) = \neg(p \to \neg p)$$

In this way we obtain the following three cases:

(i) if $p=1$, then $Np=1$;
(ii) if $p=\frac{1}{2}$, then $Np=0$;
(iii) if $p=0$, then $Np=0$.

Again, the second and third cases are quite plain, whereas the first case is questionable and will be discussed in a moment.[1] Note that in L2 you could *not* distinguish among p, Mp, Np: all these truth-functions are in classical semantics logically equivalent, so that in L2 you would have a real modal collapse in connection with this truth-functional characterizations of the operators 'M' and 'N' given above. What makes the difference, here, is clearly the third value ½.

On the basis of such a truth-functional characterizations of the modal operators Łukasiewicz shows that the classical "modal square" of oppositions results verified in Ł3, i.e. that the following propositions are Ł3-tautologies:

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[1] A definition for the operator «it is *impossible* that...» can be trivially provided along the same lines.

1. $Np \leftrightarrow \neg M \neg p$
2. $Mp \leftrightarrow \neg N \neg p$
3. $\neg(Np \wedge \neg Mp)$
4. $Mp \vee M\neg p$

Further, Łukasiewicz shows that in Ł3 the most representative modal principles of the logical and philosophical tradition (and the principle logically derivable from them) can be proved to be valid. According to him, they are:

(A) *Ab oportere ad esse valet consequentia*;
(B) *Unumquodque, quando est, oportet esse*;
(C) For some p, it is possible that p and it is possible that not-p.

The first principle is representative of the relations among ontological modalities. The second one has to do with "temporal necessity". The third one asserts the existence of some contingent event. As a matter of fact, if we give the modal principles (A)-(C) the following formal fashion:

(a) $Np \rightarrow p$
(b) $p \rightarrow (p \rightarrow Np)$
(c) $\exists p\ (Mp \wedge M\neg p)$

it comes out that they are Ł3-tautologies.[2] On the contrary, in L2 no monadic propositional operator does exist which verifies (a)-(c) together; further, in bivalent semantics they give rise either to modal collapse – as we've seen – or to contradiction (for a detailed proof see Łukasiewicz [1931]).[3]

Let's now turn back to the problematic cases concerning the value of some modal propositions; as we have seen, if p=0, then Mp=0: if p *happens* to be false, the fact expressed by p is impossible. And the other case: if p=1, then Np=1: if p *happens* to be true, the fact expressed by p is necessary. It follows that if a proposition happens to be true or false, it is *not* contingent. It follows from that that no proposition which is true or false can be said to be contingent. This sounds quite counterintuitive. Factual truth and falsity seem not to imply necessity and impossibility!

Right, if you have in mind *logical* modalities (ex. true in every / some interpretation of the propositional variables; or derivable from logical axioms my means of rules of inference). But here logical modalities are not at issue. Rather, Łukasiewicz is concerned with *ontological* modalities, i.e. different ways in which events can occur in the world. If a proposition expressing a future event happens to be now definitely true, that event cannot not occur:[4] it is already determined and in *this* sense the

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[2] Principle (c) is immediately verified by the very assumption of the existence of a third logical value corresponding to possible events.

[3] Note that in L2 you cannot even *assert* the principle (b), because of the classical validity of the 'contraction law' $((\alpha \rightarrow (\alpha \rightarrow \beta)) \leftrightarrow (\alpha \rightarrow \beta)$, which fails in Ł3.

[4] According to Łukasiewicz, a proposition p is true at a given instant t if and only if something in the world occurs at t which makes the proposition at issue true. Now, if p concerns a future event *and* p is

proposition expressing it is a necessary one. Analogously, if a proposition expressing a future event happens to be now definitely false, that event cannot occur: its non-existence is determined in advance. Note that this point does not amount to the logical equivalence between 'true' and 'necessary' on one side and 'false' and 'impossible' on the other: in Ł3 $p \leftrightarrow Np$ and $\neg p \leftrightarrow \neg M \neg p$ are *not* tautologies!

Along the lines of Prior [1953] we can finally observe that in this context the fundamental opposition concerns the set of events which are already determined – both in a "positive" and in a "negative" way – and the set of events which are still undetermined, i.e. opened to alternative possibilities (the "realm" of contingent events). And the system Ł3 worked out by Łukasiewicz fits admirably to this way of looking at the matter.

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true at present, this means that now some event already exists which is the *cause* of the future event expressed by p; and this future event follows *necessarily* from its cause.