

On the left of Beppo Levi?

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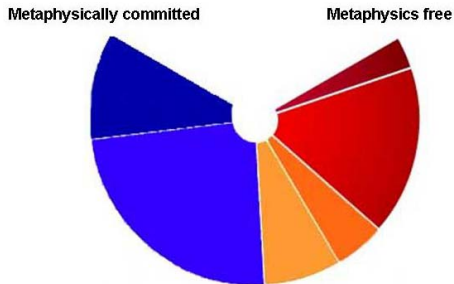
A political prologue

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K. Gödel 1961: left- vs. right-wing foundational views.

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2. Levi's principle in modern terms

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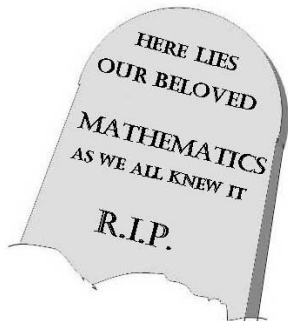
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 - 2.1 Alternatives

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2. Levi's principle in modern terms
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3. AP at work

The choice “mystery”

The victim



The assassin

The assassin

AC If X, Y are sets, and $S \subseteq X \times Y$ is such that for every $a \in X$ there exists $b \in Y$ such that $(a, b) \in S$, then there exists a function $f : X \rightarrow Y$ such that, for all $a \in X$, $(a, f(a)) \in S$.

The assassin

The assumption of Zermelo's [choice] postulate contradicts the very nature of analysis and it must be then rejected as deprived of any sense.

[B. Levi, Riflessioni sopra alcuni principii della teoria degli aggregati e delle funzioni, 1918]

The investigation begins

Giuseppe Peano (1858–1932)



G.P., Démonstration de l'intégrabilité des équations différentielles ordinaires, Math. Ann. 37, 1890

Our P.E.

Beppo Levi (1875–1961)



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Beppo Levi (1875–1961)



B.L., *Intorno alla teoria degli aggregati*, R. Ist. Lomb. di Sc. e Lett., 1902

Levi 1902: the partition principle

(dalla dissertazione di F. Bernstein, 1901)

Lemma

Let \mathfrak{F} be a family of non-empty and disjoint sets. Then the cardinal number of $A = \bigcup \mathfrak{F}$ is greater, or equal to the cardinal number of \mathfrak{F} .

Choice by approximation

*Exhuming what was “buried
in the graveyard of dead languages”*

On deductive domains

Any given mathematical argument presupposes [...] one or more aggregates, for each of which IT IS POSTULATED the possibility of picking an arbitrary element, as a prime and irreducible act of thought.

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- ▶ **Prime aggregate**: a collection of math. objects for which the above picking act is postulated.
- ▶ **Deductive Domain**: the collection of all prime aggregates.

On deductive domains

Levi's assumption and the existence property: deductive domains provide an answer to the "demand of rigour of the so-called intuitionism".

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Levi's assumption and the existence property: deductive domains provide an answer to the "demand of rigour of the so-called intuitionism".

[E]very existential proof is also a constructive one in a conveniently made precise deductive domain.

[B. Levi, La nozione di "dominio deduttivo" e la sua importanza in taluni argomenti relativi ai fondamenti dell'analisi, 1934]

Functions vs. Operations

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2. A *deductive domain* is a family Ω of prime aggregates for which there exists a function $f_\Omega : \Omega \rightarrow \bigcup \Omega$, such that $f_\Omega(A) \in A$ for every $A \in \Omega$.
3. For a given deductive domain Ω , a *natural extension* of it is a family of prime aggregates Ω' with $\Omega \subseteq \Omega'$, for which there exists a function $f_{\Omega'} : \Omega' \rightarrow \bigcup \Omega'$ such that (i) $f_{\Omega'}(A') \in A'$ ($A' \in \Omega'$), and (ii) $f_{\Omega'}(A) = f_\Omega(A)$ ($A \in \Omega$).

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Definition

Given $X, Y \neq \emptyset$ sets, F is an *operation from X to Y* (in symbols: $F : X \rightsquigarrow Y$) if, for every $a \in X$, $\emptyset \neq F(a) \subseteq Y$.

Functions vs. Operations

AC_{Op} For X, Y sets, and $F : X \rightsquigarrow Y$ operation, there exists a *(total) function* $f : X \rightarrow Y$ such that, for all $a \in X$, $f(a) \in F(a)$.

The Approximation Principle (AP)

AP Let Ω be a deductive domain, Y a (prime) aggregate outside it, $F : \Omega \rightsquigarrow Y$ operation. Let further $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$ be a function such that, for every $X, X' \subseteq Y$, $d(X, X') = 0$ if and only if $X = X'$. Let $A \in \Omega$. If

$$\begin{aligned} & (\forall \delta > 0)(\exists x_1, \dots, x_n \in A)(\forall A', A'' \in \Omega) \\ & (x_1, \dots, x_n \in A' \cap A'' \rightarrow d(F(A'), F(A'')) < \delta) \end{aligned}$$

then, $\Omega' := \Omega \cup \{F(A)\}$, is a *natural extension* of Ω .

AP as a choice principle

Let us refer the displayed condition of AP, as a formula $\psi(A, F, d)$.
Then one has

Fact

AP is equivalent to the fact that for a given deductive domain Ω , Y aggregate, and $F : \Omega \rightsquigarrow Y$ operation, there exists a partial function $f^ : \Omega \rightarrow Y$ such that for every $A \in \Omega$, $\psi(A, F, d) \Rightarrow f^*(A) \in F(A)$.*

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PROOF. Let Ω , Y and F be given as in the above statement. Then AP yields that $(\Omega' := \Omega \cup B, f')$ is a deductive domain, where $B = \{F(A) \mid A \in \Omega, \psi(A, F, d)\}$. By definition, we have then that there exists $f' : \Omega' \rightarrow \bigcup \Omega'$ with $f'(A) \in A$, $A \in \Omega'$. Let $A \in \Omega$ be among the elements of Ω such that $\psi(A, F, d)$ holds. Set $f^*(A) = f'(F(A))$ and the theorem follows. The other direction of the equivalence is a simple application of previous definitions. ■

AP as an approximation principle

Let us now go back to the original condition of AP:

$$\begin{aligned} & (\forall \delta > 0)(\exists x_1, \dots, x_n \in A)(\forall A', A'' \in \Omega) \\ & (x_1, \dots, x_n \in A' \cap A'' \rightarrow d(F(A'), F(A'')) < \delta) \end{aligned}$$

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It can be rephrased as saying: *$F(A)$ is an acceptable collection to pick elements from, if every pairs of 'finite approximations' of it are close, in the sense of d , one to the other (namely, no matter how little a $\delta \in \mathbb{R}^+$ is chosen, every such pair of elements $A', A'' \in \Omega$ is such that $d(F(A'), F(A'')) < \delta$).*

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Fact

Given a deductive domain Ω , Y aggregate, $F : \Omega \rightsquigarrow Y$ operation, and $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$ such that, for every $X, X' \subseteq Y$, $d(X, X') = 0$ iff $X = X'$, then:

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1. $AC_{\mathbb{N}}$ implies that for every A such that $\psi(A, F, d)$ holds, there exist a function f such that $f(A) \in F(A)$.

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1. $AC_{\mathbb{N}}$ implies that for every A such that $\psi(A, F, d)$ holds, there exist a function f such that $f(A) \in F(A)$.
2. $AC_{\mathbb{N}}$ implies that, if $A \in \Omega$ is such that $\psi(A, F, d)$ holds, there exists a countable set $C \subseteq A$ such that for all $A' \in \Omega$ with $C \subseteq A'$, it is $F(A) = F(A')$.

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Given a deductive domain Ω , Y aggregate, $F : \Omega \rightsquigarrow Y$ operation, and $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$ such that, for every $X, X' \subseteq Y$, $d(X, X') = 0$ iff $X = X'$, then:

1. $AC_{\mathbb{N}}$ implies that for every A such that $\psi(A, F, d)$ holds, there exist a function f such that $f(A) \in F(A)$.

PROOF. Let Ω, Y, F, d as in the hypothesis, and A be such that $\psi(F, A, d)$ holds.

Fix a descending sequence of positive rational numbers $\langle \delta_i \rangle_{i \in \mathbb{N}}$ such that $\lim_{i \in \mathbb{N}} \delta_i = 0$. Let $a_{1_i}, \dots, a_{n_i} \in A$ be such that, if $A', A'' \in \Omega$ and $a_{1_i}, \dots, a_{n_i} \in A' \cap A''$, then $d(F(A'), F(A'')) < \delta_i$.

Take \mathfrak{F} to be the family $\mathfrak{F} = \{ \langle a_{1_i}, \dots, a_{n_i} \rangle \}_{i \in \mathbb{N}}$, which exists by $AC_{\mathbb{N}}$.

Now, we prove a bit more than what is stated. Namely, we show that $AC_{\mathbb{N}}$ entails the existence of a function f such that, for every $A' \in \Omega$, if $a_{1_i}, \dots, a_{n_i} \in A'$ for some $i \in \mathbb{N}$, then $f(A') \in F(A')$.

Once this is given, the statement we want to prove clearly follows.

Then, define $F' : \Omega \rightsquigarrow Y$ as

$F'(i) = X \subseteq \bigcap \{ F(A') \mid a_{1_i}, \dots, a_{n_i} \in A \cap A' \}$. Then, $AC_{\mathbb{N}}$ yields a function f' such that, for every $i \in \mathbb{N}$, $f'(i) \subseteq F'(i) \subseteq F(A') \cap F(A'')$, for all $A', A'' \in \Omega$ such that $a_{1_i}, \dots, a_{n_i} \in A' \cap A''$.

Let $A' \in \Omega$ be such that, for some $i \in \mathbb{N}$, it is $a_{1_i}, \dots, a_{n_i} \in A \cap A'$. Set $f(A') = f'(\min k \in \mathbb{N} (a_{1_k}, \dots, a_{n_k} \in A \cap A'))$.

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Fact

Given a deductive domain Ω , Y aggregate, $F : \Omega \rightsquigarrow Y$ operation, and $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$ such that, for every $X, X' \subseteq Y$, $d(X, X') = 0$ iff $X = X'$, then:

2. $AC_{\mathbb{N}}$ implies that, if $A \in \Omega$ is such that $\psi(A, F, d)$ holds, there exists a countable set $C \subseteq A$ such that for all $A' \in \Omega$ with $C \subseteq A'$, it is $F(A) = F(A')$.

PROOF. Let Ω, Y, F, d as in the hypothesis, $A \in \Omega$ such that $\psi(A, F, d)$ holds. Let $\langle \delta_i \rangle_{i \in \mathbb{N}}$ be a descending sequence of positive rational numbers such that $\lim_{i \in \mathbb{N}} \delta_i = 0$.

Let $a_{1_i}, \dots, a_{n_i} \in A$ be such that, if $A', A'' \in \Omega$ and $a_{1_i}, \dots, a_{n_i} \in A' \cap A''$, then $d(F(A'), F(A'')) < \delta_i$.

Take \mathfrak{F} to be the family $\mathfrak{F} = \{\langle a_{1_i}, \dots, a_{n_i} \rangle\}_{i \in \mathbb{N}}$ (requires $AC_{\mathbb{N}}$).

It follows that, if $A' \in \Omega$ is such that $\bigcup \mathfrak{F} \subseteq A'$, it is $d(F(A'), F(A)) < \delta_i$ for every $i \in \mathbb{N}$, namely $d(F(A'), F(A)) = 0$. Hence, $F(A) = F(A')$ as required.

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- ▶ this latter fact does establish a sort of 'countable bound' over the application of AP, as the function $f_A : \{Z \supseteq A \mid \bigcup \mathfrak{F} \subseteq Z, Z \text{ countable}\} \rightarrow Y$, which exists by $AC_{\mathbb{N}}$, is already a choice function for the set $\{Z \supseteq A \mid \bigcup \mathfrak{F} \subseteq Z\}$ (provided A is such that $\psi(A, F, d)$ holds).

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However,

- ▶ this does not suffice to yield (at least, not immediately) a choice function by $AC_{\mathbb{N}}$ alone for the whole set $\{A \in \Omega \mid \psi(A, F, d)\}$.

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Nonetheless,

- ▶ Levi thought this could be important for the general appreciation of AP owing to an “unconscious tendency” of many mathematician that “consent infinitely many arbitrary choices, *in case* they are denumerably many” (hence, choice up to $AC_{\mathbb{N}}$ is safe).

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ma un tale ampliamento, mi pare si possa enunciare il seguente

Principio di approssimazione. — Sia assegnato un dominio deduttivo Ω , nel quale sia primo l'aggregato dei numeri reali (o almeno l'aggregato dei numeri razionali); siano A, B, C, \dots aggregati definiti in esso e contenuti in aggregati primi in esso (tali quindi che sia permesso considerarne un elemento arbitrario). Sia invece E un aggregato primo non appartenente al dominio deduttivo Ω , ciascun elemento del quale possa considerarsi costituito di infiniti elementi scelti in modo arbitrario negli aggregati A, B, C, \dots (eventualmente in alcuni soltanto di essi): sia assegnata una funzione $f(x)$ ⁽¹⁹⁾ rispetto alla quale il dominio D della x sia contenuto in E , mentre il corrispondente dominio F della funzione sia contenuto in un aggregato G , primo o non, appartenente o non a Ω ; sia

infine definita una funzione numerica $g(y, z)$ delle coppie di elementi di G la quale sia nulla sempre e solo quando $y = z$, supponiamo che z sia un elemento di D tale che, assegnato ad arbitrio un numero ϵ , fra gli elementi di A, B, C, \dots che costituiscono z , se ne possa trovare un numero finito n tale che, indicandoli con a', a'' due elementi qualunque di D avrete comuni con a i dati n elementi, sia sempre $d(a', a'') < \epsilon$.

Non consideriamo allora l'affermazione " $f(a)$ esiste ", come appartenente all'AMPLIAMENTO MATERIALE DEL DOMINIO DEDUTTIVO Ω . Analogamente il sostituto ad alcuni numeri

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infine definita una funzione numerica $d(y, z)$ delle coppie di elementi di G la quale sia nulla sempre e solo quando $y = z$; supponiamo che a sia un elemento di D tale che, assegnato ad arbitrio un numero b , fra gli elementi di A, B, C, \dots che costituiscono a , se ne possa fissare un numero finito n tale che, indicando con a', a'' due elementi qualunque di D aventi comuni con a i detti n elementi, sia sempre $d(f(a'), f(a'')) < b$.

Noi consideriamo allora l'affermazione " $f(a)$ esiste " come appartenente all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω .

Annlicheremo il principio ad alcuni esempi

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infine definita una funzione numerica $d(y, z)$ delle coppie di elementi di G la quale sia nulla sempre e solo quando $y=z$; supponiamo che a sia un elemento di D tale che, assegnato ad arbitrio un numero b , fra gli elementi di A, B, C, \dots che costituiscono a , se ne possa fissare un numero finito n tale che, indicando con a', a'' due elementi qualunque di D aventi comuni con a i detti n elementi, sia sempre $d(f(a'), f(a'')) < b$.

Noi consideriamo allora l'affermazione * $f(a)$ esiste come appartenente all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω .

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infine definita una funzione numerica $f(x, y)$ della coppia di elementi di G la quale sia nulla sempre o solo quando $y = x$; supponiamo che x sia un elemento di D tale che, assegnato ad arbitrio un numero ϵ , fra gli elementi di A, B, C, \dots che costituiscono x , ce ne possa trovare un numero finito n tale che, indicando con x_n dai elementi qualunque di D costati ognuno con n dati n elementi, sia sempre $f(x_n, x_n) < \epsilon$.

Sia consideriamo allora l'affermazione * $f(x) < \epsilon$, come appartenente all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω . Anche questa si rivela un altro assioma.

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Infine definita una funzione numerica $d(y, \delta)$ della coppia di elementi di Ω in quale sia nulla sempre e solo quando $y = \delta$, e rappresenti che δ sia un elemento di Ω tale che, assegnato ad arbitrio un numero ϵ , fra gli elementi di A, B, C, \dots che costituiscono x , se ne possa trovare un numero finito n tale che, indicata con x, x' due elementi qualunque di D assai vicini tra a i fatti n elementi, sia sempre $d(f(x), f(x')) < \epsilon$.

(12) Consideriamo allora l'affermazione "In tal caso, come approssimazione all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω , Amplifichiamo il primitivo ad alcuni numeri".

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tematica, t. v, pag. 143 (1909).

(12) Alla parola "funzione", si dà qui il significato più generale (V. la mia *Introduzione all'analisi matematica*, vol. I, § III, Parma, 1916).

infer definite una funzione numerica $f(x)$ della coppia di elementi di Ω la quale sia nulla sempre e solo quando $y=0$; supponiamo che a un elemento di Ω tale che, secondo un arbitrario numero b , fra gli elementi di A, B, C, \dots che costituiscono x , si sia potuto fissare un numero finito n tale che, indicando con x', x'' due elementi qualunque di Ω diversi ammetti con n dati n elementi, sia sempre $d(f(x'), f(x'')) < b$.

Si considerino allora l'affermazione " $f(x)$ esiste ", come appartenente all'insieme primitivo del dominio deduttivo Ω . Analizzatela il principio ad alcuni esempi.

Are we too radical?

valori assunti da queste costituiranno il **campo di variabilità** o **dominio del sistema di funzioni**.

8. Funzioni univoche e plurivoche.— A determinati valori delle variabili può corrispondere un solo o più valori, anche infiniti, di una data funzione: così, per restare negli esempi precedenti, la funzione « padre di x » non può assumere più di un valore per un valore assegnato ad x ; ma la funzione « figlio di x » potrà invece assumere più valori per un dato valore di x . La funzione « polinomio di grado x » ammette un'infinità di valori per un dato valore di x intero positivo (non ammette valori per altri valori di x).

Una funzione si dice a **un sol valore** o **univoche** o **uniforme** o **monodroma** nel campo di variabilità delle sue variabili quando essa assume un solo valore per ciascun sistema di valori delle variabili in detto campo. In caso contrario si dice a **più valori** o **plurivoche** o **multiforme** o **polidroma**.

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Principio di approssimazione. — Sia assegnato un dominio deduttivo Ω , nel quale sia primo l'aggregato dei numeri reali (o almeno l'aggregato dei numeri razionali); siano A, B, C, \dots aggregati definiti in esso e contenuti in aggregati primi in esso (tali quindi che sia permesso considerarne un elemento arbitrario). Sia invece E un aggregato primo non appartenente al dominio deduttivo Ω , ciascun elemento del quale possa considerarsi costituito di infiniti elementi scelti in modo arbitrario negli aggregati A, B, C, \dots (eventualmente in alcuni soltanto di essi); sia assegnata una funzione $f(x)$ ($^{\circ}$) rispetto alla quale il dominio D della x sia contenuto in E , mentre il corrispondente dominio F della funzione sia contenuto in un aggregato G , primo o non, appartenente o non a Ω ; sia

infine definita una funzione numerica $d(y, z)$ delle coppie di elementi di G la quale sia nulla sempre e solo quando $y=z$: supponiamo che a sia un elemento di D tale che, assegnato ad arbitrio un numero b , fra gli elementi di A, B, C, \dots che costituiscono a , se ne possa fissare un numero finito n tale che, indicando con a', a'' due elementi qualunque di D aventi comuni con a i detti n elementi, sia sempre $d(f(a'), f(a'')) < b$.

Noi consideriamo allora l'affermazione " $f(a)$ esiste " come appartenente all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω .

Annlicheremo il principio ad alcuni esempi

Are we too radical?

Definition

Given two non-void sets X, Y , we say that F is a (pointwise) operation from X to Y (in symbols: $F : X \rightsquigarrow Y$) if, for every $a \in X$, $F(a) \in Y$ (we put: $out(F, a) = \{b \in Y \mid b = F(a)\}$).

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AP_{POp} Let Ω be a deductive domain, Y an aggregate outside it, $F : \Omega \rightsquigarrow Y$ a pointwise operation. Let further $d : Y \times Y \rightarrow \mathbb{Q}_0^+$ be a function such that $d(x, y) = 0$ iff $x = y$.
Let $A \in \Omega$. If

$$(\forall \delta > 0)(\exists x_1, \dots, x_n \in A)(\forall A', A'' \in \Omega) \\ (x_1, \dots, x_n \in A' \cap A'' \rightarrow d(out(F, A'), out(F, A'')) < \delta)$$

(where $d(out(F, A'), out(F, A''))$ is a shorthand for $(\forall y \in out(F, A'))(\forall z \in out(F, A''))(d(y, z) < \delta)$), then $\Omega' := \Omega \cup \{out(F, A)\}$, is a *natural extension* of Ω .

Are we too radical?

Then, it is possible to prove

PROPOSITION

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PROOF. From left to right, given $F : \Omega \rightsquigarrow Y$, define $F' : \Omega \rightsquigarrow Y$ as $F'(X) := out(F, X)$, for every $X \in \Omega$. Define a function $d' : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$, by

$$d'(X, X') = \begin{cases} 0, & \text{if } X = X' \\ \sup\{d(x, x') \mid x \in X, x' \in X'\} & \end{cases}$$

Then, clearly $X = X'$ entails $d(X, X') = 0$. Viceversa, if $X \neq X'$, then there are $x \in X, x' \in X'$ such that $d(x, x') > 0$. But then, by definition, $d'(X, X') \geq d(x, x') > 0$. Then one can easily argue that if $A \in \Omega$ is such that the AP_{POp} -condition is satisfied, the original AP-condition is satisfied as well, and, by applying AP, one gets the result.

Are we too radical?

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PROOF. Viceversa, from right to left, we are given an operation $F : \Omega \rightsquigarrow Y$, and a function $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \rightarrow \mathbb{Q}_0^+$. Define $F' : \Omega \rightsquigarrow Y$ to be such that, for every $A \in \Omega$, $\text{out}(F', A) \subseteq F(A)$. Further, for $x \in Y$, put $\mathcal{P}_x(Y) := \{Z \subseteq Y \mid x \in Z\}$. Then, let $d' : Y \times Y \rightarrow \mathbb{Q}_0^+$ be defined by

$$d'(x, y) = \begin{cases} 0, & \text{if } x = y \\ \sup\{d(Z, Z') \mid Z \in \mathcal{P}_x(Y), Z' \in \mathcal{P}_y(Y)\} & \end{cases}$$

Then again, $x = y$ entails $d'(x, y) = 0$. If, on the other hand, $x \neq y$, then $d'(x, y) \geq d(\{x\}, \{y\}) > 0$.

The theorem follows by reasoning exactly as for the other direction. ■

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Non consideriamo allora l'affermazione " $f(a)$ esiste " come appartenente all'AMPLIAMENTO NATURALE DEL DOMINIO DEDUTTIVO Ω . Anzichè cerchiamo il principio ad alcuni esempi

AP at work

The Heine–Borel theorem

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1. A metric space (X, d_X) is *compact* if every open cover of it has a finite subcover.
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Then, one has:

Theorem (Heine–Borel)

The Euclidean space (\mathbb{R}^n, d_E) is strongly compact.

A generalisation to open(*) sets

Definition

1. A metric space (X, d_X) is *Lindelöf* if every open cover of it has a countable subcover.
2. A metric space (X, d_X) is *strongly Lindelöf** if every open *and bounded* subset of it is Lindelöf.

A generalisation to open(^{*}) sets

Lemma (E. Borel 1917, B. Levi 1918)

Let A be an open and bounded set of reals. Fix $r^* > 0$, and let $A^* = \{a \in A \mid (\exists r \geq r^*) B_r(a) \subseteq A\}$. Then, it is possible to prove:

1. A^* has a finite cover C^* , namely

$$(\exists a_1, \dots, a_n \in A^*)(\exists r_1, \dots, r_n \geq r^*) A^* \subseteq \bigcup_{1 \leq i \leq n} B_{r_i}(a_i) \quad (1)$$

2. Furthermore, C^* is such that any two members of it have their center laying outside the other circle (in symbols: for $B_{r_i}(a_i), B_{r_j}(a_j) \in C^*$, then

$$a_i \neq a_j \rightarrow a_i \notin B_{r_j}(a_j) \wedge a_j \notin B_{r_i}(a_i) \quad (2)$$

is the case).

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PROOF. Let $A \subseteq \mathbb{R}^2$ be an open, bounded set whatsoever. Let $\langle r_n \rangle_{n \in \mathbb{N}}$ be a descending sequence of positive rationals such that $\lim_{n \in \mathbb{N}} r_n = 0$. Let $\mathfrak{F} = \{C_n\}_{n \in \mathbb{N}}$ be the corresponding family (which exists by $AC_{\mathbb{N}}$), such that, for each $n \in \mathbb{N}$, C_n is the finite cover of $A_n = \{a \in A \mid (\exists r \geq r_n) B_r(a) \subseteq A\}$ which exists by the previous Lemma. Put $\mathfrak{F}' = \{\bigcup_{n \in \mathbb{N}} C_n\}$. Then,

$$A = \bigcup \mathfrak{F}'$$

which ends the proof. ■

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Lemma

Let A be an open and bounded set, $\langle r_n \rangle_{n \in \mathbb{N}}$ a descending sequence of positive rationals with $\lim_{n \in \mathbb{N}} r_n = 0$, and, for every $n \in \mathbb{N}$, $A_n = \{a \in A \mid (\exists r \geq r_n) B_r(a) \subseteq A\}$. Let further $\mathfrak{F} = \{\langle a_{1_i}, \dots, a_{m_i} \rangle\}_{i \in \mathbb{N}}$ be the family of those sets of finite elements, whose existence is ensured by the previous lemma, such that, for every $i \in \mathbb{N}$, there exists $r_{1_i}, \dots, r_{m_i} > 0$ with

$$A_i \subseteq \bigcup_{1 \leq k \leq m} B_{r_{k_i}}(a_{k_i})$$

Take $\Omega = \mathcal{P}_{\aleph_0}(A)$. Finally, take Y to be the set of sets X 's of (radii of) circles with center in a , not containing the center of any other circle which is (whose radius is) in X itself.

Then one proves, by AP_{POp} , that there exists a function $f : \{Z \in \Omega \mid \bigcup \mathfrak{F} \subseteq Z\} \rightarrow Y$.

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Then one proves, by AP_{POp}, that there exists a function $f : \{Z \in \Omega \mid \bigcup \mathfrak{F} \subseteq Z\} \rightarrow Y$.

Notice that once such a function f were available, one would have in particular

$$A \subseteq \bigcup \{B_r(a) \mid a \in \bigcup \mathfrak{F} \wedge r \in f(\bigcup \mathfrak{F})\}$$

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However, this depends upon $AC_{\mathbb{N}}$ (which was unproblematic for Levi).

A general result

Definition

1. $A \subseteq X$ is *dense in* (X, d_X) if for every $b \in X$, and for every $\epsilon > 0$, there exists $a \in A$ such that $d_X(b, a) < \epsilon$.
2. A metric space (X, d_X) is *separable* if there exists $A \subseteq X$ which is dense in (X, d_X) .

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Then, it is possible to prove a satisfactorily general result, namely:

Theorem

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PROOF. 1. Let $S \subseteq X$ be countable and dense in (X, d_X) . Let $A \subseteq X$ be open. Then $S \cap A$ is dense in A . For, by openness of A , for all $a \in A$ there exists $r > 0$ such that $B_r(a) \subseteq A$. By S being dense in (X, d_X) , for every $\epsilon \leq r$ there exists $s \in S$ such that $d_X(a, s) < \epsilon \leq r$, that is $s \in A$.

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PROOF. 2. Put $S_A := S \cap A$, $A \subseteq X$ open set, S dense in (X, d_X) . By openness of A one has that an operation $F : S_A \rightsquigarrow \mathbb{R}^+$ exists, such that, for every $s \in S_A$, and $r \in F(s)$, one has $r > 0$ and $B_r(s) \subseteq A$. Now, we want to prove, by AP, that there exists a function $f^* : S_A \rightarrow \mathbb{R}^+$ such that, for every $s \in S_A$, $B_{f(s)}(s) \subseteq A$. We try to prove a bit more, namely that this function is such that the $f(s)$'s ($s \in S_A$) are 'big enough' in the following sense:

$$\forall a \in (A \setminus \{s\}) \forall r > 0 (B_r(a) \subseteq A \wedge s \in B_{\frac{r}{2}}(a) \rightarrow f(s) \geq \frac{r}{2}) \quad (1)$$

Now, this makes sense since, for $a \in (A \setminus \{s\})$, $r > 0$, and $s \in B_{\frac{r}{2}}(a) \subseteq B_r(a) \subseteq A$, we have $B_{\frac{r}{2}}(s) \subseteq B_r(a) \subseteq A$. That is, by openness of A we can equally assume an operation $F' : S_A \rightsquigarrow \mathbb{R}^+$ exists, such that, for every $s \in S_A$, $r \in F'(s)$ iff $r > 0$, $B_r(s) \subseteq A$, and the condition (1) above is satisfied.

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PROOF. Let $F'' : \mathcal{P}_1(S_A) \rightsquigarrow \mathbb{R}^+$ (where $\mathcal{P}_1(S_A)$ is the set of singletons of elements of S_A) be defined by $F''(\{a\}) = F(a)$, $a \in S_A$. Set $\Omega = \mathcal{P}_1(S_A)$, and put $d : \mathcal{P}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{R}^+) \rightarrow \mathbb{Q}_0^+$ be defined as, for every $X, X' \subseteq \mathbb{R}^+$:

$$d(X, X') = \sup\{r \mid r = 0 \vee r \in (X \cup X') \setminus (X \cap X')\}$$

Clearly, $d(X, X') = 0$ iff $(X \cup X') \setminus (X \cap X') = \emptyset$ iff $X = X'$. This then gives, in particular, $d(X, X) = 0$ for every $X \in \mathcal{P}(\mathbb{R}^+)$. Now, for $X = \{a\}, X' \in \mathcal{P}_1(S_A)$ it is $a \in X \cap X'$ iff $X = X'$, and since, as we noticed, $d(F(X), F(X')) = 0 < \delta$, the condition $\psi(X, F'', d)$ is satisfied in this trivial sense for every $X \in \mathcal{P}_1(S_A) = \Omega$. Then, AP implies that there exists a function $f : \mathcal{P}_1(S_A) \rightarrow \mathbb{R}^+$ such that, for every $s \in S_A$, $f(\{s\}) \in F''(\{a\}) = F(a)$. Set finally $f^*(s) = f(\{s\})$, $s \in S_A$.

A general result

Then, it is possible to prove a satisfactorily general result, namely:

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AP implies that every separable metric space (X, d_X) is strongly Lindelöf.

PROOF. 3. Fix $A \subseteq X$ arbitrary open and bounded set.
Assume thus $a \in A$ and $r > 0$ such that $B_r(a) \subseteq A$.
From 1 we know that there exists $s \in S_A$ such that $d_X(a, s) < r/2$. That is, $s \in B_{r/2}(a) \subseteq B_r(a) \subseteq A$.
But then, 2 entails $a \in B_{\frac{r}{2}}(s) \subseteq B_{f(s)}(s) \subseteq A$.
Hence, $C = \{B_{f(s)}(s) \mid s \in S_A\}$ is the sought-for countable cover of A . ■

Conclusion

AP and the foundations of mathematics

AP and the foundations of mathematics

Levi's approach to the problem of foundations *via* deductive domain was *not* a 'left-wing' one:

Ella caratterizza il Suo pensiero coll'aforisma: In principio è il segno; io potrei forse cercare di riassumere analogamente il mio dicendo che: In principio è l'aggregato; però la differenza non è, credo, che nelle parole e nella forma. . .

[B.L., Sui procedimenti transfiniti (Auszug auf einem Briefe an Herrn Hilbert), Math. Ann., 1923]

AP and the foundations of mathematics

This fact notwithstanding, Levi aimed to present deductive domains as a tool for clarifying what the foundational quarrels amounted to (see his letter to Hilbert, 1923).

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- ▶ Given a set of mathematical statements, to determine the most restricted deductive domain which makes them provable.

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Relatedly, he then proposed to investigate systematically the following two problems (1934):

- ▶ Given a set of mathematical statements, to determine the most restricted deductive domain which makes them provable.
- ▶ Given a deductive domain, to determine the set of statements which are made provable by it.

A *natural* application of AP?

While working with AP, one cannot avoid a sense of ad-hocness and artificiality, mainly due to the way in which a deductive domain, and the distance function d must be chosen somehow arbitrarily.

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Then, one may ask

Is there a significant application of AP, which is natural, at least in the sense that it holds true independently from the choice of the function d ?

The end