On the proof strength of quasi–inductive definitions

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Workshop on Proof Theory & Constructivism
Leeds, *July, 2009*

On quasi–induction

The quasi–inductive schema

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A *quasi–inductive sequence* $\langle \Gamma_\alpha \mid \alpha \in ON \rangle$ of sets $\Gamma_\alpha \subseteq \mathbb{N}$, is the one given by the clauses

$$\begin{aligned} \Gamma_0 &= \emptyset \\ \Gamma_{\alpha+1} &= \Gamma(\Gamma_\alpha) \\ \Gamma_\lambda &= \liminf_{\beta \to \lambda} \Gamma_\beta, \ \lambda \text{ limit} \end{aligned}$$

with $\liminf_{\beta \to \lambda} \Gamma_\beta := \{n \mid \exists \alpha < \lambda \forall \beta < \lambda (\alpha \leq \beta \to n \in \Gamma_\beta)\}$

Basic properties

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- for $(\Gamma_\infty^+, \Gamma_\infty^-)$ *stability pair* of the sequence of sets, with

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  one shows that there are (limit) *stabilization ordinals* δs, for which $\Gamma_\delta = \Gamma^+_\infty$ and $\Gamma^-_\delta = \liminf_{\alpha\to\delta}(\mathbb{N}\setminus\Gamma_\alpha) = \Gamma^-_\infty$;

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- levels as such appear again and again in the quasi–inductive iteration of the operator, according to a given 'period'. That is, for a stabilization ordinal δ one proves that

$$\exists\beta\forall\gamma[\Gamma_{\delta+\beta\gamma} = \Gamma_\delta]$$

Axioms

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Language: operator forms

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- the notion of formula for the expanded language must be re–defined so to comprise atoms $\mathcal{H}_A(n, \alpha)$ (abbrev. $n \in \mathcal{H}_A^\alpha$) with $n \in TERM^N$ and $\alpha \in TERM^\Omega$.

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Language: conventions

Let:

$$
\begin{aligned}
(\mathcal{H}_A^\alpha \equiv \mathcal{H}_A^\beta) &:\equiv \forall x(x \in \mathcal{H}_A^\alpha \leftrightarrow x \in \mathcal{H}_A^\beta) \\
x \in \mathcal{H}_A^+(\infty) &:\equiv \exists\beta\forall\delta(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^\delta) \\
x \in \mathcal{H}_A^-(\infty) &:\equiv \exists\beta\forall\delta(\beta \leq \delta \rightarrow x \notin \mathcal{H}_A^\delta) \\
x \in \mathcal{H}_A^\lambda &:\equiv \exists\beta < \lambda\forall\delta < \lambda(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^\delta) \\
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Axioms

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Axioms: logical, arithmetical, ordinal–theoretical

For a fixed $\mathbf{K} = \Delta_n, \Pi_n, \Sigma_n$, the axioms of QID($\mathbf{K}$) amount at:

- axioms of *first–order predicate logic with equality*
- the *axioms of arithmetic* (with CI)
- standard *assumptions on the ordering* $<_\Omega$, on *ordinal individual constants*, the *defining equations* of the stock of primitive ordinal functions as well as *axioms on their basic properties* (monotonicity, inverses), plus a schema of *transfinite induction*

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Axioms: QID

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(QID.1) $x \in \mathcal{H}_A^0 \rightarrow x \neq x$

(QID.2) $x \in \mathcal{H}_A^{\alpha+1} \leftrightarrow A(x, \mathcal{H}_A^\alpha)$

(QID.3) $Lim(\lambda) \rightarrow [x \in \mathcal{H}_A^\lambda \leftrightarrow (\exists \alpha < \lambda)(\forall \beta < \lambda)(\alpha \leq \beta \rightarrow x \in \mathcal{H}_A^\beta)]$

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(QID.4) $\forall \alpha \exists \lambda (Lim(\lambda) \wedge \alpha < \lambda \wedge (\mathcal{H}_A^+(\lambda) \equiv \mathcal{H}_A^+(\infty)) \wedge (\mathcal{H}_A^-(\lambda) \equiv \mathcal{H}_A^-(\infty)))$

Some results

Periodicity

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PROPOSITION

Let σ be any limit ordinal such that $\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{+\infty}$ and $\mathcal{H}_A^{-\sigma} \equiv \mathcal{H}_A^{-\infty}$, for a **K***–operator form $A(x, X)$. Then QID(***K***) proves that there exists a unique ordinal $p(\sigma) > 0$, the period of σ, such that:*

(i) *for every ordinal γ, $\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{\sigma + p(\sigma)\gamma}$*

(ii) *for every ordinal $\alpha > \sigma$ there exists an ordinal $0 \leq \nu < p(\sigma)$ such that $\mathcal{H}_A^{\alpha} \equiv \mathcal{H}_A^{\sigma+\nu}$*

Lower bound: *Are QIDs a next natural step?*

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FID(K)

(OP.1) $\mathcal{P}^{\alpha}_{A}(s) \leftrightarrow \mathcal{P}^{<\alpha}_{A}(s) \vee A(s, \mathcal{P}^{<\alpha}_{A})$

(OP.2) $A(s, \mathcal{P}^{\infty}_{A}) \rightarrow \mathcal{P}^{\infty}_{A}(s)$

[where $\mathcal{P}^{<\alpha}_{A}(s) := (\exists \beta < \alpha)\mathcal{P}^{\beta}_{A}(s)$, and $\mathcal{P}^{\infty}_{A}(s) := \exists \beta \mathcal{P}^{\beta}_{A}(s)$]

Lower bound: *Are QIDs a next natural step?*

PROPOSITION

Theories $\mathsf{FID}(\mathbf{K})$ *can be embedded into theories* $\mathsf{QID}(\mathbf{K})$.

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(Set–theoretic) Upper bound

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PROPOSITION

The theory for arithmetical QIDs, $\mathbf{QID}(\Pi_\infty)$, *is embeddable in* $\mathbf{T}$.

(Set–theoretic) Upper bound

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$$\mathrm{QH}_A(\alpha, f) := \begin{cases} Fun(f) \wedge dom(f) = \alpha \wedge \\ \wedge(\forall\beta < \alpha)[(\beta = 0 \wedge f(\beta) = 0) \vee \\ \vee(\exists\gamma < \alpha)(\beta = \gamma + 1 \wedge f(\beta) = \{z \in \mathrm{N} \mid A^{\mathrm{N}}(z, f(\gamma))\}) \vee \\ \vee(Lim(\beta) \wedge f(\beta) = \bigcup_{\gamma<\beta} \bigcap_{\gamma\leq\delta<\beta} f(\delta))] \end{cases}$$

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$$x \in H_A^\alpha \quad := \quad \exists f[\mathrm{QH}_A(\alpha + 1, f) \wedge x \in f(\alpha)]$$

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$$\begin{aligned} x \in H_A^{\alpha} &:= \exists f[\mathrm{QH}_A(\alpha + 1, f) \wedge x \in f(\alpha)] \\ x \in H_A^{-\alpha} &:= (\exists \beta < \alpha)(\forall \gamma < \alpha)(\beta \leq \gamma \rightarrow x \notin H_A^{\gamma}) \\ x \in H_A^{+\infty} &:= \exists \beta \forall \gamma(\beta \leq \gamma \rightarrow x \in H_A^{\gamma}) \\ x \in H_A^{-\infty} &:= \exists \beta \forall \gamma(\beta \leq \gamma \rightarrow x \notin H_A^{\gamma}) \end{aligned}$$

(Set–theoretic) Upper bound

Lemma

For all operator forms $A(x, X)$, KP proves:

1. $\forall\alpha\exists f \mathrm{QH}_A(\alpha, f)$.
2. $\mathrm{QH}_A(\alpha, f) \wedge \beta < \alpha \rightarrow \mathrm{QH}_A(\beta, f)$
3. $\mathrm{QH}_A(\alpha, f) \wedge \mathrm{QH}_A(\beta, g) \wedge \alpha \leq \beta \rightarrow (\forall\gamma < \alpha)(f(\gamma) = g(\gamma))$
4. $n \in \mathrm{N} \rightarrow (n \in H_A^{\alpha+1} \leftrightarrow A^{\mathrm{N}}(n, H_A^{\alpha}))$
5. $n \in \mathrm{N} \wedge Lim(\lambda) \rightarrow (n \in H_A^{\lambda} \leftrightarrow (\exists\alpha < \lambda)(\forall\beta < \lambda)(\alpha \leq \beta \rightarrow n \in H_A^{\beta}))$

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5. $n \in \mathrm{N} \wedge Lim(\lambda) \rightarrow (n \in H_A^{\lambda} \leftrightarrow (\exists\alpha < \lambda)(\forall\beta < \lambda)(\alpha \leq \beta \rightarrow n \in H_A^{\beta}))$

Stages H_A^{α}s can be equivalently described by means of the Π_1 condition

$$x \in H_A^{\alpha} \leftrightarrow \forall f[\mathrm{QH}_A(f, \alpha + 1) \rightarrow x \in f(\alpha)]$$

Hence, formulas $x \in H_A^{\alpha}$ are Δ_1^{T}, while $x \in H_A^{+\infty}, x \in H_A^{-\infty}$ are both Σ_2^{T}.

(Set–theoretic) Upper bound

SECOND: use Π_2–collection.

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PROPOSITION (Covering)

In T it is provable that, for every ordinal α, there exists a limit ordinal $\delta > \alpha$ such that $H_A^{+\infty} \subseteq H_A^{\delta}$, $H_A^{-\infty} \subseteq H_A^{-\delta}$, $H_A^{\delta} \cap H_A^{-\infty} = \emptyset$ and $H_A^{-\delta} \cap H_A^{+\infty} = \emptyset$.

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(Set–theoretic) Upper bound

THIRD: use Σ_2–collection.

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PROPOSITION (Stability)

In T it is provable that, for every arithmetical operator form $A(x, X)$, $\forall\alpha\exists\lambda(\alpha < \lambda \wedge H_A^{\lambda} \equiv H_A^{+\infty} \wedge H_A^{-\lambda} \equiv H_A^{-\infty})$.

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Final Remarks

Future work

- Upper bound refinement (P. Welch: use Σ_2–**KP**)

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- Is this an *exact* bound?

The end

Ordinal axioms

(Ω.1) $\forall \alpha\beta(\alpha = \beta \vee \alpha < \beta \vee \beta \vee \alpha)$

(Ω.2) $\forall \alpha(\neg\alpha < \alpha)$

(Ω.3) $\forall \alpha\beta\gamma(\alpha < \beta \wedge \beta < \gamma \to \alpha < \gamma)$

(Ω.4) $\forall \alpha(0_\Omega \leq \alpha)$ [where $\alpha \leq \beta := (\alpha < \beta \vee \alpha = \beta)$]

(Ω.5) $\forall \alpha(\alpha < \alpha')$ [with $\alpha' = \mathit{succ}_\Omega(\alpha)$]

(Ω.6) $\forall \alpha\beta(\alpha < \beta \to \alpha' \leq \beta)$

(Ω.7) $0_\Omega < \omega \wedge \forall \alpha < \omega(\alpha' < \omega)$

(Ω.8) $\forall \lambda(\mathit{Lim}(\lambda) \to \omega \leq \lambda)$

[where $\mathit{Lim}(\alpha) := (0 < \alpha \wedge \forall \beta < \alpha(\beta' < \alpha))$]

Ordinal axioms

$$(\Omega.9) \quad \forall\alpha(\alpha + 0_\Omega = \alpha)$$

$$(\Omega.10) \quad \forall\alpha\beta(\alpha + \beta' = (\alpha + \beta)')$$

$$(\Omega.11) \quad \forall\alpha\beta\gamma(\alpha < \beta \to \gamma + \alpha < \gamma + \beta)$$

$$(\Omega.12) \quad \forall\alpha\beta\gamma(\alpha \leq \beta \to \alpha + \gamma \leq \beta + \gamma)$$

$$(\Omega.13) \quad \forall\alpha(\alpha 0_\Omega = 0_\Omega \alpha = 0_\Omega)$$

$$(\Omega.14) \quad \forall\alpha\beta(\alpha\beta' = \alpha\beta + \alpha)$$

$$(\Omega.15) \quad \forall\alpha\beta\gamma(0_\Omega < \gamma \wedge \alpha < \beta \to \gamma\alpha < \gamma\beta)$$

$$(\Omega.16) \quad \forall\alpha\beta\gamma(\alpha \leq \beta \to \alpha\gamma \leq \beta\gamma)$$

$$(\Omega.17) \quad \forall\alpha\beta(\alpha < \beta \to \exists\gamma \leq \beta(\alpha + \gamma = \beta))$$

$$(\Omega.18) \quad \forall\alpha\beta(0_\Omega < \beta \to \exists\gamma \leq \alpha\exists\delta < \beta(\alpha = \beta\gamma + \delta))$$

Ordinal axioms

$$(\mathcal{L}(\mathbf{K}) - \mathsf{I}_N) \quad A(0) \wedge \forall x(A(x) \rightarrow A(x')) \rightarrow \forall x A(x)$$

$$(\mathcal{L}(\mathbf{K}) - \mathsf{I}_\Omega) \quad \forall \alpha((\forall \beta < \alpha)A(\beta) \rightarrow A(\alpha)) \rightarrow \forall \alpha A(\alpha)$$

◀ B

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3. Define the embedding in the expected manner with $\mathcal{P}_A^{\alpha} \longmapsto \mathcal{H}_B^{\alpha}$.

◀ B

Proof of the Covering Lemma

Since

$$(\forall x \in \mathsf{N})\exists\beta(x \in H_A^{+\infty} \rightarrow (\forall\gamma \geq \beta)(x \in H_A^{\gamma}))$$

is a simple consequence of the definitions, (COLL) ensures then that

$$\exists b(\forall x \in \mathsf{N})(\exists\beta \in b)(x \in H_A^{+\infty} \rightarrow (\forall\gamma \geq \beta)(x \in H_A^{\gamma})) \quad (1)$$

By (SEP), put $b' = \{\beta \in b \mid (\exists n \in \mathsf{N})(\forall\gamma \geq \beta)(n \in H_A^{\gamma})\}$.

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Find sets c, c' playing for $H_A^{-\infty}$ the role b and b' play for $H_A^{+\infty}$.
Finally, let α be any ordinal. Take δ to be the least limit ordinal such that $\xi < \delta$ where $\xi = \alpha \cup b' \cup c'$. It's easy to see that δ satisfies the lemma. Q.E.D.

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3. This is used for defining a Δ_2 recursive enumeration F of W, with every $x \in W$ occurring infinitely often;
4. By (Σ_2–)recursion again, one defines a sequence $\langle H_A^\beta \mid \beta < \mu \rangle$, with μ limit, $H_A^0 = H_A^\delta$ and if $x \in W$ then x is not 'stable below μ';

Proof of the Stability Lemma: Overview

1. Ordinals δs are ‘almost good’, except that they may contain elements outside $H_A^{+/-\infty}$;
2. The set $W = \{x \in \mathrm{N} \mid U_\delta(x)\}$ of them (w.r.t. to a fixed δ), admits a Σ_1–definition;
3. This is used for defining a Δ_2 recursive enumeration F of W, with every $x \in W$ occurring infinitely often;
4. By (Σ_2–)recursion again, one defines a sequence $\langle H_A^\beta \mid \beta < \mu \rangle$, with μ limit, $H_A^0 = H_A^\delta$ and if $x \in W$ then x is not ‘stable below μ’;
5. By definition then, $H_A^\delta \subseteq H_A^\mu$ and $x \in W$ entails $x \notin H_A^\mu$.

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Proof of the Stability Lemma: Details

Notice that if δ is an ordinal given by covering, we have that

$$\begin{aligned} x \in H_A^{+\infty} &\leftrightarrow \forall\beta(\delta \leq \beta \rightarrow x \in H_A^{\beta}) \\ x \in H_A^{-\infty} &\leftrightarrow \forall\beta(\delta \leq \beta \rightarrow x \notin H_A^{\beta}) \end{aligned}$$

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This motivates the following (Σ_1-)definition of unstable elements:

Definition

For every $n \in \mathrm{N}$ and δ arbitrary but fixed ordinal given by covering, we say that *n is unstable (relatively to δ)* (abbreviated: $U_\delta(n)$) if

$$U_\delta(n) := \exists \beta(\delta \leq \beta \wedge n \in H_A^{\beta}) \wedge \exists \gamma(\delta \leq \gamma \wedge n \notin H_A^{\gamma})$$

Proof of the Stability Lemma: Details

We first need a function enumerating $W = \{n \in \mathrm{N} \mid U_\delta(n)\}$:

$$
\begin{array}{lcl}
F(0) & := & \min_N z \in \mathrm{N}. U_\delta(z) \\
F(\alpha) & := & \begin{cases} \min_N z \in \mathrm{N}. U_\delta(z) \wedge (\forall \beta < \alpha)(F(\beta) < z), & \text{if it exists} \\ F(0), & \text{otherwise} \end{cases}
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We have:

- W is a Σ_1 set, the totality of F simply follows by the Σ_2–recursion theorem.
- F exhausts W in ω steps, and it keeps exhausting it after every ξ number of steps afterwards, for ξ limit ordinal. Elements of W occurr *infinitely often* in a list provided by F if we set $dom(F) = \gamma$, where γ is such that $Lim^+(\gamma) := 0 < \gamma \wedge (\forall \alpha < \gamma)(\exists \beta < \gamma)(\alpha < \beta \wedge Lim(\beta))$

Proof of the Stability Lemma: Details

Let then λ be an ordinal such that $Lim^{+}(\lambda)$ holds. We define:

$$\begin{array}{rcl} G(0) & = & \delta \\ G(\alpha+1) & = & \begin{cases} \min \mu . G(\alpha) < \mu \wedge F(\alpha) \in H_A^{\mu}, & \text{if } F(\alpha) \notin H_A^{G(\alpha)} \\ \min \mu . G(\alpha) < \mu \wedge F(\alpha) \notin H_A^{\mu}, & \text{otherwise} \end{cases} \\ G(\xi) & = & \min \mu . \sup(\{G(\beta) \mid \beta < \xi\}) < \mu, \ \xi \text{ limit} \end{array}$$

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- μ_0 is a limit ordinal satisfying the property given by the covering lemma;
- G is strictly increasing below λ (hence, below m_0);
- members of W behave as unstable elements, hence they are not retained at $H_A^{\mu_0}, H_A^{-\mu_0}$. Q.E.D.