Logic Colloquium 2004
Turin, 25/7 – 31/7

A note on Gödel's philosophy of mathematics in the light of his *Nachlass*

Riccardo Bruni
Dept. of Philosophy
University of Florence
riccardobruni@hotmail.com

1. Gödel's philosophy: the received view.

Gödel's Doctrine (GD – S. Feferman, 1987), mathematical and philosophical formulations:

GDm the transfinite iteration of the power-set operation is necessary to account for finitary mathematics;

GDp abstract concepts (i.e., notions which belong to areas of mathematical reasoning like set theory, where we don't exercise intuition at its basic levels), are necessary for domains (like arithmetic) where, on the contrary, a more 'concrete' form of intuition is at work.

(A) Gödel's heuristic path toward incompleteness (historical remark):

> "The true reason for incompleteness . . . is that the formation of ever higher types can be continued into the transfinite . . . while in any formal system at most denumerably many of them are available" (*1931*, note 48a).

(A) Evidences supporting Feferman's interpretation from Gödel's correspondence:

1. the attempt to fulfil Hilbert's expectations of a consistency proof for analysis within (finitary) arithmetic, led Gödel to formalize arithmetically a truth predicate for arithmetical formulas (see Gödel's corr. with P. Bernays – *CW4*, pp. 90–105, letters 3–5 – with J. van Heijenoort – *CW5*, p. 313, lett. 6 – and with J. Balas – *CW4*, pp. 9-11);
2. paradoxes arising from such an attempt (i.e., background required by Tarski's theorem), brought him to formulate a non-constructive version of the incompleteness theorems (see corr. with Zermelo – *CW5*, p. 426–428, lett. 2 – or *1934*, §7).

(B) A general version of the incompleteness theorems:

> "That [axioms of infinity] have consequences also far outside the domain of very great transfinite numbers ... can be proved; each of them ... can ... be shown to increase the number of decidable propositions even in the field of Diophantine equations" (*1947*, p. 182).

2.1 GD: relation with the incompleteness theorems.

(B) Gödel's unpublished undecidability theorems (*CW3*, pp. 164–175):

1. There's no mechanical procedure for deciding the class $\mathcal{A}$ of mathematical problems of the form:

   $$\forall \vec{x}\, \exists \vec{y}\, P(\vec{x}, \vec{y}) = 0$$

   where P is a polynomial with integer coefficients.

2. In every formal system, in the language of which every problem of the class $\mathcal{A}$ is expressible (and which is moreover correct with respect to $\mathcal{A}$), there exist undecidable problems belonging to $\mathcal{A}$.

Gödel's most stable position toward GCH reads (see, e.g., *1947*):

- Cantor's conjecture is *false*;
- it is *independent* from ZF(C);
- GCH can be probably decided (refuted) by means of axioms which "a more profound understanding of the concepts underlying logic and mathematics" would enable us to recognize as implied by ordinary set theoretical axioms or axioms of infinity as known at that time.

2.2 GD: reflections on Cantor's Continumm Hypothesis.

Gödel and axioms of infinity (see *CW3*, pp. 305–307):

1. if mathematics is reduced to set theory (i.e., if an uncritical standpoint toward mathematical reasoning is assumed), and if we try to axiomatize the concept of set in a stepwise manner, we're naturally led to admit transfinitely always stronger assumptions about the largeness of the universe of all sets;

2. these principles provably have consequences outside the domain to which they primarily belong, and

3. (although indirectly) they may help us to solve presently unsolvable set theoretical or analytical problems.

Gödel on position 3 above:

> "It is not impossibile that for such a concept of demonstrability [i.e., demostrability from ordinary set theoretical axioms plus axioms of infinity, non constructively and generally characterized,] some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets" (*1946*, p. 151).

3.1 Gödel's Doctrine and philosophy: Gödel's rationalism.

Gödel's *epistemological* assumption:

- the power of mathematical reasoning based on a relation with abstract terms may be unended (it is possible that no *absolutely unsolvable* mathematical problem exist, that is to say, it is possible that no *mathematical fact* lie beyond our capacity of perceving the world of mathematical entities).

Absolutely unsolvable mathematical problems related to GCH and axiom of constructibility, had been previously mentioned by Gödel himself (see *CW3*, p. 129, p. 175, pp. 184–185).

Gödel's approach to human mind:

"What Turing disregards completely [in his 1937 paper] is the fact that *mind in its use, is not static, but constantly developing*, i.e. that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding. ... Therefore, although at each stage the number and precision of the abstract terms at our disposal may be *finite*, both ... may *converge toward infinity* in the course of the application of the [mental] procedure" (*1972a*, p. 306).

3.2 Gödel's Doctrine and philosophy: Gödel's mentalism.

1. the class $\mathcal{A}$ of Π_2^0 diophantine problems is a bound for the abilities of any Turing machine in the domain of mathematics;
2. a transfinite sequence of axioms of infinity stemming from concepts on which mathematics (set theory) is based and which cannot be described constructively, allows us to the solve problems belonging to the class $\mathcal{A}$;
3. *if an uncritical viewpoint toward mathematics is assumed* and, consequently, *if we look at the transfinite sequence of axioms in question as a natural result of mathematical reasoning*, human mind seems to be always capable of new assumptions (produced non-constructively) leading it to surpass the abilities of a Turing machine.

4. Gödel's contender: Turing on logic, machines and mind.

According to Turing (*1939*, §11) mathematical reasoning is made possible by two faculties:

intuition the ability of recognizing a mathematical proposition as true without any conscious train of thoughts.

ingenuity that allows to build *proofs* of intuitive judgements (constructions where intuition is needed at less doubtful levels).

Turing concludes that:

- (despite negative results like Gödel's theorems) *sensible* metamathematical programs still can be pursued *non-constructively*, in order to make clear when a step is *purely mechanical*, and when it is *purely intuitive*, while

- either the 'pre-Gödel' ideal that it is possible to eliminate intuition by means of a finite number of rules, or the theorical assumption that ingenuity is "available in unlimited supply", shouldn't be pursued any longer.

Turing's program for an intelligent machinery (see, in particular, his *1947* and *1948* papers):

1. *Fallibility*: as negative results in the logical field indicates, no machine can be infallible *and* intelligence. But no conclusion can be drawn from these theorems for a machine which is allowed to make mistakes (this is the essence of Turing's answer to the *mathematical objection* to the possibility of a machinery with intelligence).

2. *Education*: machines must be constructed in order to be trained, exactly in the same way a human being is subject, from infancy onward, to an appropriate educational process.

5. Turing's late contributions.

Turing's program and his view in its most radical formulation (see *1948*, pp. 120–121, *1950*, pp. 454–460):

- the infant, or, more precisely, its brain cortex, may resemble an unorganized machine (a machine built without any specific purpose);

- due to the small number of mechanisms contained in an infant's mind, it wouldn't be difficult to produce a 'child machine' mimicking the behaviour of a man at his initial state;

- education would thus consist in turning this machine into a perfectly disciplined one (i.e., a machinery which obeys to any kind of order and is capable of permorming different tasks).

Intelligence *cannot* be reduced, in Turing's opinion, to the ability of obeying orders:

> "If the untrained infant's mind is to become an intelligent one, it must acquire both *discipline* and *initiative*. So far we have been considering only discipline. To convert a brain or machine into a universal machine is the extremest form of discipline. ... But discipline is certainly not enough to produce intelligence. That which is required in addition we call initiative. ... Our task is to discover the *nature* of this residue and to *try and copy* it in machines" (*1948*, p. 125).

6. Turing vs Gödel?

- Human reasoning may have non-mechanical components: to *ingenuity* there correspond *intuition* (which cannot be contained in a finite set of rules); to *discipline* (the "extremest" form of which is identified with the behaviour of a Turing machine) there correspond *ingenuity* (the *nature* of which is left undecided).

- Human mind is not equivalent to a Turing machine: Turing's late purpose is that of building a machine (with different properties than the thoretical model) behaving *as if* it were intelligent; if such a machinery were built, he would probably think that admitting it to really have intelligence is *safer* (although not unproblematic) than denying it.

- Turing's approach to the problem of the nature of human mind is quite different than Gödel's: to the committment of the latter with a philosophical standpoint, there corresponds a choice of the former in favor of a scientific research, the goals of which are largely independent from the solution of the conceptual problem.