

A language and axioms for quasi-inductive definitions

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Taming circularity

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This definition is **circular** in a straightforward sense, hence **illegitimate** (i.e., it is not possible to give a ‘meaning’ – read: *classical* extension – to the predicate G).

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3. fix a *rule* to *single out regularities* (i.e. *recurring elements/hypotheses*), obtained by iterating 1-2.

► Views

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$$[x = a \vee (x = b \wedge \neg G(x)) / x \notin X]$$

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Input		Output
\emptyset	\Rightarrow	$\{a, b\}$
$\{a, b\}$	\Rightarrow	$\{a\}$
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$\{b\}$	\Rightarrow	$\{a\}$
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The limit rule and why we need it

Let instead G be T , and our definition be

$$T(x) := \left. \begin{array}{l} (x = \ulcorner s = t \urcorner \wedge \text{val}(s) = \text{val}(t)) \\ \vee (x = \ulcorner T(s) \urcorner \wedge T(\ulcorner T(s) \urcorner)) \\ \vee (x = \ulcorner \neg \varphi \urcorner \wedge \neg T(\ulcorner \varphi \urcorner)) \\ \dots \end{array} \right\} \varphi(x, T)$$

Namely: let $\varphi(x, T)$ be the arithmetical formula defining a Tarskian truth predicate.

The limit rule and why we need it

Furthermore, let \mathbf{F} be the set:

$$\mathbf{F} := \{\varphi_0, \varphi_1, \dots, \varphi_n, \dots\}$$

with:

$$\begin{aligned}\varphi_0 &:= \top \\ \varphi_{n+1} &:= T\varphi_n\end{aligned}$$

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PROBLEM: cannot conclude $\mathbf{F} \subseteq T$ by finitary means.

'Alternative' views

Herzberger's (the 'pure' theory)	Gupta's (the 'corrected' th.)	Belnap's (the 'maximal' th.)
\emptyset	$Z \subseteq \mathbb{N}$	$Z \subseteq \mathbb{N}$
hyperarithmetic	=	=
liminf	liminf $\cup h_0$	$\{X \mid X \text{ coherent}^\dagger\}$

$^\dagger X \subseteq \mathbb{N}$ s. t., for λ limit

$$h_{<\lambda}^+ \subseteq X \quad X \cap h_{<\lambda}^- = \emptyset$$

with $h_{<\lambda}^{+/-}$ sets of positive/negative stable elements (below λ).

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QID's 'mining'

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QID

$$\begin{aligned} \Gamma_0 &= \emptyset \\ \Gamma_{\alpha+1} &= \Gamma(\Gamma_\alpha) \\ \lambda \text{ limit, } \Gamma_\lambda &= \liminf_{\beta \rightarrow \lambda} \Gamma_\beta = \\ &= \{x \mid \exists \alpha < \lambda \forall \beta < \lambda (\alpha \leq \beta \rightarrow x \in \Gamma_\beta)\} \end{aligned}$$

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with $(\Gamma_\infty^+, \Gamma_\infty^-)$

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Axioms and (some) results

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- ▶ the collection $TERM_0^\Omega$ is the least containing $0_\Omega, \omega$ and ind. var. of the second sort, and which is closed under the given ordinal functions;
- ▶ the collection $FORM_0$ contains *atomic formulas* $n = m, \alpha = \beta, \alpha < \beta$ (with $n, m \in TERM_0^N, \alpha, \beta \in TERM_0^\Omega$), and is *closed under boolean logical operations and quantification on both sorts of variables*.

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- ▶ the notion of ‘formula’ for the expanded language must be re-defined so to comprise **atoms** $\mathcal{H}_A(n, \alpha)$ (abbrev. $n \in \mathcal{H}_A^\alpha$) with $n \in \text{TERM}^N$ and $\alpha \in \text{TERM}^\Omega$.

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Language: conventions

Let:

$$(\mathcal{H}_A^\alpha \equiv \mathcal{H}_A^\beta) \quad ::= \quad \forall x (x \in \mathcal{H}_A^\alpha \leftrightarrow x \in \mathcal{H}_A^\beta)$$

$$x \in \mathcal{H}_A^{+\infty} \quad ::= \quad \exists \beta \forall \delta (\beta \leq \delta \rightarrow x \in \mathcal{H}_A^\delta)$$

$$x \in \mathcal{H}_A^{-\infty} \quad ::= \quad \exists \beta \forall \delta (\beta \leq \delta \rightarrow x \notin \mathcal{H}_A^\delta)$$

$$x \in \mathcal{H}_A^{+\lambda} \quad ::= \quad (\exists \beta < \lambda) (\forall \delta < \lambda) (\beta \leq \delta \rightarrow x \in \mathcal{H}_A^\delta)$$

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Axioms

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Axioms: logical, arithmetical, ordinal–theoretical

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- ▶ axioms of *first–order predicate classical logic with equality*
- ▶ the *axioms of arithmetic* (with CI included)
- ▶ standard *assumptions on the ordering* $<$, on *ordinal individual constants*, the *defining equations* of the stock of primitive ordinal functions as well as *axioms on their basic properties* (monotonicity, inverses), plus a schema of *transfinite induction*

▶ Details

▶ QID

Ordinal axioms: details

$$(\Omega.1) \quad \forall \alpha \beta (\alpha = \beta \vee \alpha < \beta \vee \beta < \alpha)$$

$$(\Omega.2) \quad \forall \alpha (\neg \alpha < \alpha)$$

$$(\Omega.3) \quad \forall \alpha \beta \gamma (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma)$$

$$(\Omega.4) \quad \forall \alpha (0_\Omega \leq \alpha) \quad [\text{where } \alpha \leq \beta := (\alpha < \beta \vee \alpha = \beta)]$$

$$(\Omega.5) \quad \forall \alpha (\alpha < \alpha') \quad [\text{with } \alpha' = \text{succ}_\Omega(\alpha)]$$

$$(\Omega.6) \quad \forall \alpha \beta (\alpha < \beta \rightarrow \alpha' \leq \beta)$$

$$(\Omega.7) \quad 0_\Omega < \omega \wedge \forall \alpha < \omega (\alpha' < \omega)$$

$$(\Omega.8) \quad \forall \lambda (\text{Lim}(\lambda) \rightarrow \omega \leq \lambda)$$

$$[\text{where } \text{Lim}(\alpha) := (0 < \alpha \wedge \forall \beta < \alpha (\beta' < \alpha))]$$

Ordinal axioms: details

- ($\Omega.9$) $\forall \alpha (\alpha + 0_\Omega = \alpha)$
- ($\Omega.10$) $\forall \alpha \beta (\alpha + \beta' = (\alpha + \beta)')$
- ($\Omega.11$) $\forall \alpha \beta \gamma (\alpha < \beta \rightarrow \gamma + \alpha < \gamma + \beta)$
- ($\Omega.12$) $\forall \alpha \beta \gamma (\alpha \leq \beta \rightarrow \alpha + \gamma \leq \beta + \gamma)$
- ($\Omega.13$) $\forall \alpha (\alpha 0_\Omega = 0_\Omega \alpha = 0_\Omega)$
- ($\Omega.14$) $\forall \alpha \beta (\alpha \beta' = \alpha \beta + \alpha)$
- ($\Omega.15$) $\forall \alpha \beta \gamma (0_\Omega < \gamma \wedge \alpha < \beta \rightarrow \gamma \alpha < \gamma \beta)$
- ($\Omega.16$) $\forall \alpha \beta \gamma (\alpha \leq \beta \rightarrow \alpha \gamma \leq \beta \gamma)$
- ($\Omega.17$) $\forall \alpha \beta (\alpha < \beta \rightarrow \exists \gamma \leq \beta (\alpha + \gamma = \beta))$
- ($\Omega.18$) $\forall \alpha \beta (0_\Omega < \beta \rightarrow \exists \gamma \leq \alpha \exists \delta < \beta (\alpha = \beta \gamma + \delta))$

Ordinal axioms: details

$$(\mathcal{L}(\mathbf{K}) - I_N) \quad A(0) \wedge \forall x(A(x) \rightarrow A(x')) \rightarrow \forall x A(x)$$

$$(\mathcal{L}(\mathbf{K}) - I_\Omega) \quad \forall \alpha((\forall \beta < \alpha) A(\beta) \rightarrow A(\alpha)) \rightarrow \forall \alpha A(\alpha)$$

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$$\text{(QID.1)} \quad x \in \mathcal{H}_A^{0_\Omega} \rightarrow x \neq x$$

$$\text{(QID.2)} \quad x \in \mathcal{H}_A^{\alpha+1} \leftrightarrow A(x, \mathcal{H}_A^\alpha)$$

$$\text{(QID.3)} \quad \text{Lim}(\lambda) \rightarrow [x \in \mathcal{H}_A^\lambda \leftrightarrow (\exists \alpha < \lambda)(\forall \beta < \lambda)(\alpha \leq \beta \rightarrow x \in \mathcal{H}_\beta^A)]$$

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$$(QID.4) \ \forall \alpha \exists \lambda (Lim(\lambda) \wedge \alpha < \lambda \wedge (\mathcal{H}_A^{+\lambda} \equiv \mathcal{H}_A^{+\infty}) \wedge (\mathcal{H}_A^{-\lambda} \equiv \mathcal{H}_A^{-\infty}))$$

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- (i) for every ordinal γ , $\mathcal{H}_A^\sigma \equiv \mathcal{H}_A^{\sigma+p(\sigma)\gamma}$
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PROOF: standard



Results (II): lower bound

Theories for first-order nonmonotone inductive definitions $\text{FID}(\mathbf{K})$ are defined in a similar manner as our $\text{QID}(\mathbf{K})$ (with relations $\mathcal{P}_A^\alpha(n)$ for an operator form $A(x, X)$ whatsoever), except that the operator axioms are

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$\text{FID}(\mathbf{K})$

$$\text{(OP.1)} \quad \mathcal{P}_A^\alpha(s) \leftrightarrow \mathcal{P}_A^{<\alpha}(s) \vee A(s, \mathcal{P}_A^{<\alpha})$$

$$\text{(OP.2)} \quad A(s, \mathcal{P}_A^\infty) \rightarrow \mathcal{P}_A^\infty(s)$$

[where $\mathcal{P}_A^{<\alpha}(s) := (\exists \beta < \alpha) \mathcal{P}_A^\beta(s)$, and $\mathcal{P}_A^\infty(s) := \exists \beta \mathcal{P}_A^\beta(s)$]

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Step 1 ► $A(x, X) \xrightarrow{\text{w.l.g.}} B(x, X) := (\mathbf{X}(x) \vee A(x, X))$

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Step 2 ▶ Verify axioms (OP.1-2) for levels $\mathcal{H}_B^{\alpha/+ \infty}$ and inflationary $B(x, X)$'s (using *inclusivity*: $C(s) \rightarrow B(s, C)$ and *stability*)

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Step 1 ▶ $A(x, X) \xrightarrow{\text{w.l.g.}} B(x, X) := (X(x) \vee A(x, X))$

Step 2 ▶ Verify axioms (OP.1-2) for levels $\mathcal{H}_B^{\alpha/+ \infty}$ and inflationary $B(x, X)$'s (using *inclusivity*: $C(s) \rightarrow B(s, C)$ and *stability*)

Step 3 ▶ Define the embedding in the expected manner, namely with $\mathcal{P}_A^\alpha \mapsto \mathcal{H}_B^\alpha$

Results (III): upper bound

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GOAL

What is the theory T extending KP needed to prove the stabilization property as it is embodied by our axiom (QID.4)?

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GENERAL STRATEGY

Direct formalization of standard arguments

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$$\text{QH}_A(\alpha, f) := \left\{ \begin{array}{l} \text{Fun}(f) \wedge \text{dom}(f) = \alpha \wedge \\ \wedge (\forall \beta < \alpha) [(\beta = 0 \wedge f(\beta) = 0) \vee \\ \vee (\exists \gamma < \alpha) (\beta = \gamma + 1 \wedge f(\beta) = \{z \in \mathbb{N} \mid A^{\mathbb{N}}(z, f(\gamma))\})] \vee \\ \vee (\text{Lim}(\beta) \wedge f(\beta) = \bigcup_{\gamma < \beta} \bigcap_{\gamma \leq \delta < \beta} f(\delta))] \end{array} \right.$$

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$$\begin{aligned} x \in H_A^\alpha &:= \exists f[\text{QH}_A(\alpha + 1, f) \wedge x \in f(\alpha)] \\ x \in H_A^{-\alpha} &:= (\exists \beta < \alpha)(\gamma < \alpha)(\beta \leq \gamma \rightarrow x \notin H_A^\gamma) \\ x \in H_A^{+\infty} &:= \exists \beta \forall \gamma(\beta \leq \gamma \rightarrow x \in H_A^\gamma) \\ x \in H_A^{-\infty} &:= \exists \beta \forall \gamma(\beta \leq \gamma \rightarrow x \notin H_A^\gamma) \end{aligned}$$

Results (III): upper bound

Lemma

For all operator forms $A(x, X)$, KP proves:

1. $\forall \alpha \exists f \text{QH}_A(\alpha, f)$.
2. $\text{QH}_A(\alpha, f) \wedge \beta < \alpha \rightarrow \text{QH}_A(\beta, f)$
3. $\text{QH}_A(\alpha, f) \wedge \text{QH}_A(\beta, g) \wedge \alpha \leq \beta \rightarrow (\forall \gamma < \alpha)(f(\gamma) = g(\gamma))$
4. $\text{QH}_A(\alpha, f) \wedge \alpha < \beta \rightarrow \exists! g(\text{QH}_A(\beta, g))$
5. $n \in \mathbb{N} \rightarrow (n \in H_A^{\alpha+1} \leftrightarrow A^{\mathbb{N}}(n, H_A^\alpha))$
6. $n \in \mathbb{N} \wedge \text{Lim}(\lambda) \rightarrow (n \in H_A^\lambda \leftrightarrow (\exists \alpha < \lambda)(\forall \beta < \lambda)(\alpha \leq \beta \rightarrow n \in H_A^\beta))$

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SECOND: use collection.

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Apply it to

$$\begin{aligned} (\forall x \in \mathbf{N}) [& \exists \beta \forall \gamma (\beta \leq \gamma \rightarrow \exists f (\text{QH}_A(\gamma + 1, f) \wedge x \in f(\gamma))) \rightarrow \\ & \rightarrow \exists ! \nu (\forall \delta (\nu \leq \delta \rightarrow \exists g (\text{QH}_A(\delta + 1) \wedge x \in g(\delta))) \wedge \\ & \wedge \forall \xi (\forall \mu (\xi \leq \mu \rightarrow \exists h (\text{QH}_A(\mu + 1, h) \wedge x \in h(\mu)) \rightarrow \\ & \rightarrow \nu \leq \xi))] \end{aligned}$$

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Apply it to

$$(\forall x \in \mathbb{N})(x \in H_A^{+\infty} \rightarrow \exists! \mu (\mu = \min \xi . \forall \eta (\xi \leq \eta \rightarrow x \in H_A^\eta)))$$

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Lemma (Covering)

In $KP+(\Sigma_3\text{-COLL})$ it is provable that, for every ordinal α , there exists a limit ordinal δ such that $H_A^{+\infty} \subseteq H_A^\delta$, $H_A^{-\infty} \subseteq H_A^{-\delta}$, $H_A^\delta \cap H_A^{-\infty} = \emptyset$ and $H_A^{-\delta} \cap H_A^{+\infty} = \emptyset$.

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[Notice that ordinals δ 's given by covering are *almost good*]

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Take the function enumerating them given by

$$F(0) := \min_N z \in \mathbb{N}. U_\delta(z)$$

$$F(\alpha) := \begin{cases} \min_N z \in \mathbb{N}. U_\delta(z) \wedge (\forall \beta < \alpha)(F(\beta) < z), & \text{if it exists} \\ F(0), & \text{otherwise} \end{cases}$$

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Use (for the 1st time) (Δ_3 -SEP) (and (Π_3 -COLL)) in order to prove that F always yields a value (standard argument).

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Use $(\Delta_3\text{-SEP})$ and $(\Pi_3\text{-COLL})$ as before *plus* properties of λ and F , in order to prove that the definition ‘works’.

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By definition of G , for a limit ordinal ν such that

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By definition of G , for a limit ordinal ν such that

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it is *never* the case for an $x \in \mathbb{N}$ with $U_\delta(x)$, that it behaves as a stabel element below ν .

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Hence, since an ordinal as such can be found by collection, separation and foundation, we have in fact proved that

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Theorem (Stabilization)

In $T = KP \cup \{(\Delta_3 - SEP), (\Pi_3 - COLL)\}$ it is provable that, for every arithmetical operator form $A(x, X)$,
$$\forall \alpha \exists \lambda (\alpha < \lambda \wedge H_A^\lambda \equiv H_A^{+\infty} \wedge H_A^{-\lambda} \equiv H_A^{-\infty}).$$

Results (III): upper bound

FOURTH: define the embedding

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FOURTH: define the embedding (straightforward).

Perspective work

Short-term goal

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SHARPEN THE PROOF-THEORETIC ANALYSIS

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SHARPEN THE PROOF-THEORETIC ANALYSIS [AFA?]

- ▶ S3 SUB
- ▶ Long term

Another bound

Another way to provide an interpretation for the theory $\text{QID}(\Pi_\infty)$ of arithmetical quasi-inductive definitions is to use an assumption on elementary substructures of the universe of all sets.

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Namely, put:

$$x \prec_n V \equiv \text{Ord}(x) \wedge \forall z_1, \dots, z_k \in x (\varphi(z_1, \dots, z_k) \leftrightarrow \varphi^{(x)}(z_1, \dots, z_k))$$

for every Σ_n -formula $\varphi(x_1, \dots, x_k)$ with displayed free variables.

Then, assume

$$(\Sigma_3 - \text{SUB}) \quad \forall y \exists x (y \in x \wedge x \prec_3 V)$$

Another bound

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[Looks strong]

Another bound

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[Looks strong]

PROBLEM 2: Does it serve the purpose?

[Looks *too* strong]

The long term agenda

The long term agenda

CAN WE RECOVER THE PHILOSOPHY BEHIND?

▶ Short

A remark on truth

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- ▶ by a result of Halbach's, FS gets a model via the revision-theoretic machinery
- ▶ this model construction can be made explicit by means of our theories

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How to?

LOWER THE COMPLEXITY

How to?

LOWER THE COMPLEXITY
[A game-theoretic way out?]

▶ End

The long term agenda

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