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A solution to Curry and Hindley's problem on combinatory strong reduction

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Abstract It has often been remarked that the metatheory of strong reduction $\succ$, the combinatory analogue of $\beta\eta$-reduction $\twoheadrightarrow_{\beta\eta}$ in λ-calculus, is rather complicated. In particular, although the confluence of $\succ$ is an easy consequence of $\twoheadrightarrow_{\beta\eta}$ being confluent, no direct proof of this fact is known. Curry and Hindley's problem, dating back to 1958, asks for a self-contained proof of CR($\succ$), one which makes *no detour* through λ-calculus. We answer positively to this question, by extending and exploiting the technique of *transitivity elimination* for 'analytic' combinatory proof systems, which has been introduced in previous papers of ours. Indeed, a very short proof of CR($\succ$) immediately follows from the main result of the present paper, namely that a certain analytic proof system $\mathbf{G_e}[\mathbb{C}]$, which is equivalent to the standard proof system $\mathbf{CL_{ext}}$ of Combinatory Logic *with extensionality*, admits effective transitivity elimination. In turn, the proof of transitivity elimination — which, by the way, we are able to provide not only for $\mathbf{G_e}[\mathbb{C}]$ but also, in full generality, for arbitrary analytic combinatory systems with extensionality — employs purely proof-theoretical techniques, and is entirely contained within the theory of combinators.

Keywords Combinatory logic · extensionality · strong combinatory reduction · elimination of transitivity.

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1 Introduction

Combinatory strong reduction $\succ$ (also called *$\beta\eta$-strong reduction* in [9]) is introduced in Curry and Feys's [2], sect. 6 F, as the combinatory analogue of $\beta\eta$-reduction in λ-calculus. In other words, *strong* reduction $\succ$ is the appropriate reduction for the *extensional* theory of equality in Combinatory Logic, $\mathbf{CL_{ext}}$, just as *weak* reduction $\twoheadrightarrow_{\mathtt{w}}$

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is the appropriate reduction for the *non extensional* theory of equality in Combinatory Logic, **CL**. That is:

$$\mathbf{CL_{ext}} \vdash t = s \quad \textit{iff} \quad t \text{ goes to } s \text{ by a finite series of } \succ\text{- and reversed } \succ\text{-reductions.}$$

Strong reduction is defined by the usual rules that generate weak reduction $\twoheadrightarrow_{\mathtt{w}}$, i.e. (changing $\twoheadrightarrow_{\mathtt{w}}$ into $\succ$)

$$\frac{}{t \succ t}\,\rho \qquad \frac{}{\mathsf{K}ts \succ t}\,K \qquad \frac{}{\mathsf{S}tsr \succ tr(sr)}\,S$$

$$\frac{t \succ s}{rt \succ rs}\,\mu \qquad \frac{t \succ s}{tr \succ sr}\,\nu \qquad \frac{t \succ s \quad s \succ r}{t \succ r}\,\tau\,,$$

and the additional rule

$$\frac{s \succ r}{\lambda^* x.s \succ \lambda^* x.r}\,\xi\,,$$

where functional abstraction $\lambda^* x.t$ is defined according to the so called *strong* algorithm (having $\lambda^* x.tx := t$, if x doesn't occur in t; see sect. 4).

Actually, when dealing with strong reduction, it is preferable to take I as a *primitive* combinator besides K and S (adding $\mathsf{I}t \succ t$ to the above rules, and $\mathsf{I}t = t$ to the axiom schemas of $\mathbf{CL_{ext}}$), instead of having it defined as $\mathsf{I} := \mathsf{SKK}$, as is usually done. The motivation (see [2], § 6 F.1) is that otherwise it would follow, by the fact that $\mathsf{SK} \succ \mathsf{KI}$ does hold, that

$$\mathsf{I} \equiv \mathsf{SKK} \succ \mathsf{KIK} \succ \mathsf{K(KIK)K} \succ \ldots \quad ,$$

and so we would have a combinatory term in strong normal form ($\lambda^* x.x \equiv \mathsf{SKK}$) which is not strongly irreducible[1]. As soon as the variant with I as a primitive combinator is adopted, it holds that strongly irreducible terms are in strong normal form (*Normal Form Theorem*, first proved by Curry [2]; a shorter proof is due to Hindley and Lercher [7]), and conversely (*Converse Normal Form Theorem*, Lercher [10]).

Notwithstanding its reputation of being quite messy, much is known about the metatheory of strong reduction, see. e.g. [2] (sect. 6 F), [3] (sect. 11 E), [8] (Ch. 7), [9] (Ch. 8), [18] (§ 3.8), [1] (§ 7.2). In particular, it is known that $\succ$ has the *Church-Rosser property* ([2], sect. 6 F, Theorem 3) and that there is a (necessarily) infinite set of axiom schemes axiomatizing $\succ$ over $\twoheadrightarrow_{\mathtt{w}}$, which is recursive (Hindley [4], resp. Lercher [11]).

Let us now concentrate on CR($\succ$). Curry's proof of this result is extremely simple (see Remark 4.3), but *it is not direct*, being inferred from the corresponding result for $\lambda\beta\eta$-reduction. Hence the problem

> "c. Is it possible to prove the Church-Rosser property directly for strong reduction, without having recourse to transformations between that theory and the theory of λ-conversion? …"

[1] A combinatory term t is *strongly irreducible* iff $t \succ s$ implies $t \equiv s$, for every s. The class *SNS* of combinatory terms *in strong normal form* is defined inductively: $t \in SNS$ iff $t \equiv \lambda^* x.s$ for some $s \in SNF$, or $t \equiv xt_1 \ldots t_n$ for some variable x and $t_1, \ldots, t_n \in SNF$, with $n \geq 0$.

listed under the heading "Unsolved problems" in § 6 F.5 of [2].

A solution was advanced by Loewen in 1968 (cf. [12], [13], [14]); unfortunately, his proof seems to contain an error — as observed in Hindley's review [5] (cf. also [3], § 11 E.8). Actually, the problem is considered as still *open*, and as such it has been quite recently reproposed by Hindley [6]:

> "**Problem # 1**
> *Date:* Known since 1958!
> **Statement.** Is there a direct proof of the confluence of $\beta\eta$-strong reduction?
> **Problem Origin.** First posed by Haskell Curry and Roger Hindley.
> The $\beta\eta$-strong reduction is the combinatory analogue of $\beta\eta$-reduction in λ-calculus. It is confluent. Its only known confluence-proof is very easy ... but it depends on the having already proved the confluence of $\beta\eta$-reduction. Thus the theory of combinators is not self-contained at present. Is there a confluence proof independent of λ-calculus? "

In the present paper, we intend to provide such a confluence proof, thus avoiding the unwanted *"metabasis eis allo genos"*.

To this aim, we exploit the notion of *analytical derivation* introduced in [15], [16] and [17] for untyped equational theories of operations related to, and including, Combinatory Logic **CL** and Lambda-calculus. In particular, in the mentioned papers we introduced new equational proof systems which are equivalent to the standard 'synthetic' calculi for **CL**, $\lambda\beta$ and $\lambda\beta\eta$, and which turn out to be 'analytic' due to a main result showing (constructively) that *the transitivity rule for equality (τ) is eliminable*. As a consequence, a kind of 'subterm property' analogous to the subformula property in first-order sequent calculi admitting cut elimination does hold for transitivity-free derivations. Indeed, analytic derivations exhibit nice structural properties which can be exploited to get a non trivial streamlining of the proof-theory of the corresponding equational theories, whose consistency is now an immediate corollary of the above mentioned *subterm property*. In turn, we get applications to the theories of combinatory *weak* reduction $\twoheadrightarrow_{\mathtt{w}}$ as well as of $\lambda\beta$- and $\lambda\beta\eta$-reduction, by providing a unified framework in which new very short demonstrations of known central results like confluency, standardization, leftmost reduction strategy etc. can be given.

The transition from synthetic to analytic combinatory proof systems is essentially determined by the transformation of the combinatory axiom schemas, e.g.

$$\mathsf{S}tsr = tr(sr) \qquad [\mathrm{AX\,F}],$$

into a pair of *left* and *right* combinatory *introduction* rules, in this case

$$\frac{tr(sr)p_1 \dots p_n = q}{\mathsf{S}tsrp_1 \dots p_n = q}\,\mathsf{S}_l \qquad \textit{and} \qquad \frac{q = tr(sr)p_1 \dots p_n}{q = \mathsf{S}tsrp_1 \dots p_n}\,\mathsf{S}_r\,,$$

additionally allowing a possibly empty list $p_1 \dots p_n$ of "side terms".
As symmetry is built in the combinatory introduction rules, the *symmetry rule* for equality $[\sigma]$ is dropped in combinatory analytic proof systems, whose equality rules thus only comprise *reflexivity* restricted to atomic terms $[\rho']$, *transitivity* $[\tau]$ and *monotony*, the latter in the parallel formulation

$$\frac{t = s \qquad p = q}{tp = sq}\,App$$

replacing $[\mu]$ and $[\nu]$.

So, as far as $\mathbf{CL_{ext}}$ is concerned (and taking $\mathsf{I}, \mathsf{K}, \mathsf{S}$ as primitive combinators), the corresponding analytic proof system $\mathbf{G_e}[\mathbb{C}]$ features the "structural" rules $[\rho'], [App], [\tau]$ and the combinatory introduction rules $[\mathsf{I}_l], [\mathsf{I}_r], [\mathsf{K}_l], [\mathsf{K}_r], [\mathsf{S}_l], [\mathsf{S}_r]$. In addition, there is the *extensionality* rule $[Ext]$[2]:

$$\frac{tx = sx}{t = s}\, Ext \qquad \textit{(with } x \textit{ not occurring in } t, s\textit{)}.$$

An equivalent system $\mathbf{G}_\xi[\mathbb{C}]$ is clearly obtained by replacing $[Ext]$ by the rule $[\xi]$:

$$\frac{t = s}{\lambda^* x.t = \lambda^* x.s}\, \xi \,.$$

However, our choice of taking $[Ext]$ as the primitive extensionality rule is, in a sense, essential. First of all, because $[\xi]$ can obviously not be generalized to arbitrary combinatory systems $\mathbb{X}$, see below. Second, and more important, because $[\xi]$ is much more cumbersome than $[Ext]$, and doesn't easily lend itself to proof-theoretical analysis. To wit, as a consequence of $\mathbf{G_e}[\mathbb{C}]$ enjoying transitivity elimination (Theorem 3.4.2) we now know that also $\mathbf{G}_\xi[\mathbb{C}]$ enjoys this fundamental property (see Remark 3.4.3); however, we wonder whether and how a direct proof of this fact might be given.

The analytic proof system $\mathbf{G_e}[\mathbb{C}]$, as well as the combinatory analytic proof systems *with extensionality* $\mathbf{G_e}[\mathbb{X}]$ based on an *arbitrary* selection $\mathbb{X}$ of primitive combinators, were conjectured in [15] to admit transitivity elimination. The core of the present paper is devoted to solve this conjecture in the affirmative, by using purely proof-theoretical methods and without making any *detour* through λ-calculus. Once this task is accomplished, a direct confluence-proof of strong reduction $\succ$ is just an almost trivial corollary: we simply have to exploit the virtues of transitivity-free derivations, in particular the fact that from any given τ-free derivation of $t = s$ in $\mathbf{G_e}[\mathbb{C}]$ we can effectively extract a term r such that $t \succ r \prec s$.

The paper is organized as follows. Section 2 provides the necessary background from [15] and [17] about general *combinatory systems* $\mathbb{X}$ as well as synthetic and analytic combinatory proof systems (without and with extensionality)[3]. Section 3 is entirely devoted to the demanding proof of transitivity elimination for $\mathbf{G_e}[\mathbb{X}]$ systems, with § 3.1 providing a sort of guided tour. Finally, section 4 presents the solution of Curry and Hindley's problem.

2 Analytic Combinatory Proof-systems

2.1 Combinatory terms and combinatory systems

Let $\mathbf{X}$ be a non-empty, possibly countably infinite set of individual constants, called *primitive combinators*. By $\mathbf{T_X}$ we denote the set of all *(combinatory) terms* built over

[2] Curry calls it ζ-rule, cf. also [8]. We follow instead Barendregt's [1] usage of $[Ext]$.

[3] Apart from minor departures, we follow the terminology and the notations of [15] and [17].

$\mathbf{X}$ and the countably infinite set $\mathbf{V} = \{v_1, v_2, \dots\}$ of individual variables according to the grammar:

$$\mathbf{T_X} = \mathbf{X} \,|\, \mathbf{V} \,|\, (\mathbf{T_X T_X}) \,.$$

An *atom*, or *atomic* term, is an individual variable or a primitive combinator; all other terms are called *application* terms.

As metavariables we use, possibly with sub- or superscripts, the letters:

- F, G for primitive combinators,
- x, y, z for individual variables,
- p, q, r, s, t (occasionally d, see below) for terms,
- $\overline{p}, \overline{q}$ for (possibly empty) finite lists of terms.

The symbol $\equiv$ denotes syntactic identity between terms, which are written according to standard conventions, like association to the left. In particular, given a list $\overline{p} \equiv p_1, \dots, p_n$ with $n \geq 0$,

$$t\overline{p} \equiv \begin{cases} t, & \text{if } n = 0 \text{ (i.e., if } \overline{p} \text{ is the empty list)}, \\ (\dots((tp_1)p_2)\dots p_n), & \text{if } n > 0. \end{cases}$$

Further notational conventions and abbreviations include:

- $V(t) :=$ the set of all variables occurring in the term t;
- $\mathbf{T}^n_{\mathbf{X}} :=$ the set of all terms t such that $V(t) \subseteq \{v_1, \dots, v_n\}$ (the set of all closed terms, if $n = 0$);
- $t[x_1/s_1, \dots, x_n/s_n] :=$ the term resulting from t by simultaneous substitution of s_i for x_i $(1 \leq i \leq n)$;
- $t[s_1, \dots, s_n] := t[v_1/s_1, \dots, v_n/s_n]$;
- $\|t\| :=$ the *depth* of t, that is the maximum length of a branch in the formation tree of t, minus 1;
- $\overline{p} \sim \overline{q} :=$ the lists $\overline{p}$ and $\overline{q}$ have the same length.

Later we shall also make use of $\mathbf{X}$-*contexts*, that is terms *with some holes* $*$ in them:

$$\mathbf{H_X} = \, * \,|\, \mathbf{X} \,|\, \mathbf{V} \,|\, (\mathbf{H_X H_X}) \,.$$

We will use the letters Φ, Ψ, Θ (possibly with decorations) as metavariables for elements of $\mathbf{H_X}$. Given a context Φ and an expression E (a term t or a context Ψ), $\Phi[\![E]\!]$ shall denote the expression obtained from Φ by filling the holes, i.e. replacing all the occurrences of $*$, with E. Clearly,

$$\Phi[\![E]\!] \in \begin{cases} \mathbf{H_X}, & \text{if } E \in \mathbf{H_X}; \\ \mathbf{T_X}, & \text{if } E \in \mathbf{T_X}. \end{cases}$$

Definition 2.1.1 By a *combinatory system* $\mathbb{X}$ we mean a map, defined on a non-empty set $\mathbf{X} = dom(\mathbb{X})$ of primitive combinators, which associates to each $\mathsf{F} \in \mathbf{X}$ a pair $\langle k_{\mathsf{F}}, d_{\mathsf{F}} \rangle$ s.t.:

- k_{F}, the *index* of F under $\mathbb{X}$, is a non negative integer;
- d_{F}, the *definition* of F under $\mathbb{X}$, is a $\mathbf{T}^{k_{\mathsf{F}}}_{\mathbf{X}}$-term, i.e. $V(d_{\mathsf{F}}) \subseteq \left\{v_1, \dots, v_{k_{\mathsf{F}}}\right\}$.

Terms of the form $\mathsf{F}t_1 \dots t_{k_\mathsf{F}}$ are called $\mathbb{X}$-*redexes*. The $\mathbb{X}$-*contractum* of the redex $\mathsf{F}t_1 \dots t_{k_\mathsf{F}}$ is the term $d_\mathsf{F}[t_1,\dots,t_{k_\mathsf{F}}]$.

Notice that the case $k_\mathsf{F} = 0$ is admitted, and that a primitive combinator F may well occur in its own definition d_F, possibly together with other primitive combinators. A combinatory system $\mathbb{X}$ is said to be *linear* (*pure*) whenever, for every $\mathsf{F} \in \mathbf{X}$, d_F contains at most one occurrence of each of $v_1,\dots v_{k_\mathsf{F}}$ (resp., whenever d_F is a combination of variables)[4].

Definition 2.1.2 Given a combinatory system $\mathbb{X}$ and a primitive combinator $\mathsf{F} \in \mathbf{X}$, let:

– $[\mathrm{AX}\,\mathsf{F}]_\mathbb{X}$ be the equation schema (relatively to $\mathbf{T_X}$):

$$\mathsf{F}t_1 \dots t_{k_\mathsf{F}} = d_\mathsf{F}[t_1,\dots,t_{k_\mathsf{F}}].$$

– $[\mathsf{F}_l]_\mathbb{X}$ and $[\mathsf{F}_r]_\mathbb{X}$ be the "combinatory" inference rules

$$\frac{d_\mathsf{F}[t_1,\dots,t_{k_\mathsf{F}}]\overline{p} = s}{\mathsf{F}t_1 \dots t_{k_\mathsf{F}}\overline{p} = s}\,[\mathsf{F}_l]_\mathbb{X} \quad \text{and} \quad \frac{s = d_\mathsf{F}[t_1,\dots,t_{k_\mathsf{F}}]\overline{p}}{s = \mathsf{F}t_1 \dots t_{k_\mathsf{F}}\overline{p}}\,[\mathsf{F}_r]_\mathbb{X}\,,$$

called *left*, resp. *right* F-*introducton*.
The terms $\overline{p}$ (if any) are said to be the *side terms* of these inferences.

2.2 Synthetic and analytic combinatory proof systems

To each combinatory system $\mathbb{X}$ we now associate the 'standard' *synthetic* proof systems without / with extensionality ($\mathbf{C}[\mathbb{X}]$, resp. $\mathbf{C_e}[\mathbb{X}]$) as well as corresponding *analytic* proof systems without / with extensionality ($\mathbf{G}[\mathbb{X}]$, resp. $\mathbf{G_e}[\mathbb{X}]$).

Definition 2.2.1 $\mathbf{C}[\mathbb{X}]$ is the equational proof system determined by:

– Axiom schemas: $[\mathrm{AX}\,\mathsf{F}]_\mathbb{X},$ for each $\mathsf{F} \in \mathbf{X}$;
– Inference rules: *reflexivity* $[\rho]$, *symmetry* $[\sigma]$, *transitivity* $[\tau]$, and *monotony* $[\mu]$, $[\nu]$, i.e.

$$\frac{}{t = t}\,\rho\,, \qquad \frac{t = s}{s = t}\,\sigma\,, \qquad \frac{t = s \quad s = r}{t = r}\,\tau\,, \qquad \frac{t = s}{rt = rs}\,\mu\,, \qquad \frac{t = s}{tr = sr}\,\nu\,.$$

Definition 2.2.2 $\mathbf{G}[\mathbb{X}]$ is the equational proof system determined by:

– "Combinatory" inference rules: $[\mathsf{F}_l]_\mathbb{X}$ and $[\mathsf{F}_r]_\mathbb{X},$ for each $\mathsf{F} \in \mathbf{X}$;
– "Structural" inference rules: *reflexivity* restricted to atomic terms $[\rho']$, *transitivity* $[\tau]$ and *parallel application* $[App]$:

$$\frac{t_1 = s_1 \qquad t_2 = s_2}{t_1t_2 = s_1s_2}\,App$$

[4] Curry's terminology (cf. [2], sect. 5 C ff.) has 'proper' for 'pure', and 'not having a duplicative effect' for 'linear'.

Notice that $[\sigma]$ is not a primitive rule of $\mathbf{G}[\mathbb{X}]$.

Definition 2.2.3 $\mathbf{C_e}[\mathbb{X}]$ and $\mathbf{G_e}[\mathbb{X}]$ are the systems obtained from $\mathbf{C}[\mathbb{X}]$, respectively $\mathbf{G}[\mathbb{X}]$, by adding the extensionality rule $[Ext]$:

$$\frac{tx = sx}{t = s}\,{\scriptstyle Ext} \quad \textit{where } x \notin V(ts).$$

The variable x is called the *eigenvariable* of the inference.

Example 2.2.4 For instance, **CL** and $\mathbf{CL_{ext}}$ (variant with primitive I) coincide with $\mathbf{C}[\mathbb{C}]$, resp. $\mathbf{C_e}[\mathbb{C}]$, where $\mathbb{C}$ is the *non linear* combinatory system, defined over $\{\mathsf{I},\mathsf{K},\mathsf{S}\}$, such that:

$$\mathbb{C}(\mathsf{I}) = \langle 1, v_1\rangle,\ \ \mathbb{C}(\mathsf{K}) = \langle 2, v_1\rangle \ \text{ and } \ \mathbb{C}(\mathsf{S}) = \langle 3, v_1v_3(v_2v_3)\rangle.$$

On the other side, $\mathbf{G}[\mathbb{C}]$ and $\mathbf{G_e}[\mathbb{C}]$ coincide with the analytic versions of **CL** and $\mathbf{CL_{ext}}$ described in the Introduction.

By $\mathbf{G_{(e)}}[\mathbb{X}]$ and $\mathbf{C_{(e)}}[\mathbb{X}]$ we shall refer indifferently to $\mathbf{G}[\mathbb{X}]$ and $\mathbf{G_e}[\mathbb{X}]$, resp. $\mathbf{C}[\mathbb{X}]$ and $\mathbf{C_e}[\mathbb{X}]$. Further, we let

$$\mathbf{G^-_{(e)}}[\mathbb{X}] := \text{ the system } \mathbf{G_{(e)}}[\mathbb{X}] \textit{ minus the transitivity rule } [\tau].$$

For an arbitrary combinatory system $\mathbb{X}$, the following are easily verified (cf. [15]):

Proposition 2.2.5 *Analytic proof systems* $\mathbf{G_{(e)}}[\mathbb{X}]$ *and* $\mathbf{G^-_{(e)}}[\mathbb{X}]$ *are closed under rule* $[\sigma]$.

Proposition 2.2.6 (Equivalence) *For* $t,s \in \mathbf{T_X}$: $\ \vdash_{\mathbf{G_{(e)}}[\mathbb{X}]} t = s \iff \vdash_{\mathbf{C_{(e)}}[\mathbb{X}]} t = s.$

Proposition 2.2.7 (τ-free consistency) *If x and y are distinct variables, then* $\nvdash_{\mathbf{G^-_{(e)}}[\mathbb{X}]}$ $x = y$. *Hence* $\mathbf{G^-_{(e)}}[\mathbb{X}]$ *is consistent.*

As far as transitivity elimination is concerned, the main result of [15] is:

Proposition 2.2.8

1. *For every combinatory system* $\mathbb{X}$, $\mathbf{G}[\mathbb{X}]$ *admits (effective)*[5] τ*-elimination.*
2. *For every* linear *combinatory system* $\mathbb{X}$, $\mathbf{G_e}[\mathbb{X}]$ *admits (effective)* τ*-elimination.*

Section 3 is devoted to show that the *linearity* restriction in Proposition 2.2.8 (2) can indeed be removed. Thus, henceforth we will exclusively deal with *extensional* analytic proof systems.

[5] That is, effective provided the map $\mathbb{X}$ is effective.

2.3 Further preliminaries

Before starting the hard work, we collect here some notions, notations and auxiliary lemmas which will be needed in the rest of the paper, and are valid for any analytic extensional proof system $\mathbf{G_e}[\mathbb{X}]$.

$\mathbf{G_e}[\mathbb{X}]$-derivations, or simply *derivations* (denoted by $\mathfrak{D}, \mathfrak{E}, \ldots$), are finite binary trees which are locally correct w.r. to the inference rules $[\rho'], [App], [\tau], [\mathsf{F}_l], [\mathsf{F}_r]$ (for each $\mathsf{F} \in \mathbf{X}$) of Definition 2.2.2 and the extensionality rule $[Ext]$ of 2.2.3. Of course, we treat $[\rho']$ as a 0-premises rule by convenience. We write

$$\mathfrak{D} \vdash t = s \qquad \textit{and} \qquad \mathfrak{D} \vdash^- t = s$$

to mean that $\mathfrak{D}$ is a derivation, resp. a *transitivity free* derivation, of the equation $t = s$. Further, we conveniently indicate the final inference of a derivation $\mathfrak{D}$ by $\mathtt{end}(\mathfrak{D})$.

A derivation $\mathfrak{D}$ is said to be a *left* (*right*) derivation provided that no *right* (*left*) combinatory inference occurs in $\mathfrak{D}$. We usually write $\mathfrak{D} \vdash_L t = s$ ($\mathfrak{D} \vdash_R t = s$) to mean that $\mathfrak{D}$ is a left (right) derivation of $t = s$, and we write accordingly $\mathfrak{D} \vdash_L^- t = s$ ($\mathfrak{D} \vdash_R^- t = s$) in case $\mathfrak{D}$ is τ-free. Of a map $\mathfrak{D} \mapsto \mathfrak{D}'$ transforming derivations into derivations we say that it preserves *orientation* whenever $\mathfrak{D}'$ is a right (left) derivation provided $\mathfrak{D}$ is a right (left) derivation.

Derivations $\mathfrak{D}$ will be measured according to *size* $\mathtt{s}(\mathfrak{D})$ and *height* $\mathtt{h}(\mathfrak{D})$, and possibly according to a third parameter, *x-width* $\mathtt{w}_x(\mathfrak{D})$, the latter depending also on the variable $x \in \mathbf{V}$:

- $\mathtt{s}(\mathfrak{D}) :=$ the number of combinatory and $[Ext]$ inferences occurring in $\mathfrak{D}$;
- $\mathtt{h}(\mathfrak{D}) :=$ the maximum length (number of nodes) of a branch in $\mathfrak{D}$'s tree, minus 1;
- $\mathtt{w}_x(\mathfrak{D}) :=$ the number of occurrences of $[\rho']$-inferences of the form $x = x$ in $\mathfrak{D}$.

Due to the left / right symmetry of the combinatory introduction rules, to each derivation $\mathfrak{D}$ we may associate its *dual* derivation $\widetilde{\mathfrak{D}}$ by effecting the following transformations:

1. each node $t = s$ is replaced by $s = t$;
2. labels $[\mathsf{F}_r]$ are changed into $[\mathsf{F}_l]$ and conversely;
3. the two premises of a τ-inference, if any, are interchanged.

Obviously,

$$\mathfrak{D} \vdash t = s \quad \textit{iff} \quad \widetilde{\mathfrak{D}} \vdash s = t\,.$$

This indeed shows that $\mathbf{G_{(e)}}[\mathbb{X}]$ is closed under the symmetry rule $[\sigma]$ (cf. Proposition 2.2.5). Clearly, the transformation $\mathfrak{D} \mapsto \widetilde{\mathfrak{D}}$ preserves all the relevant parameters and properties (size, height, x-width, τ-freeness) of derivations, while it interchanges left-handedness and right-handedness.

The following lemmas on substitution will be frequently used.

Lemma 2.3.1 (Parallel substitution) *To each pair of derivations*

$$\mathfrak{D}_1 \vdash t = s \qquad \textit{and} \qquad \mathfrak{D}_2 \vdash p = q$$

and each variable $x \in V$ we can effectively associate a derivation

$$\mathfrak{D}^* \vdash t[x/p] = s[x/q]$$

which is a left (right, τ-free) derivation provided both $\mathfrak{D}_1, \mathfrak{D}_2$ are left (right, τ-free) derivations. Moreover:

$$\mathtt{s}(\mathfrak{D}^*) \leq \mathtt{s}(\mathfrak{D}_1) + \mathtt{w}_x(\mathfrak{D}_1) \cdot \mathtt{s}(\mathfrak{D}_2) \quad \textit{and} \ \mathtt{h}(\mathfrak{D}^*) \leq \mathtt{h}(\mathfrak{D}_1) + \mathtt{h}(\mathfrak{D}_2)\,.$$

Proof By straightforward induction on the height of $\mathfrak{D}_1$, taking cases according to $\mathsf{end}(\mathfrak{D})$. Note that, when considering $\mathsf{end}(\mathfrak{D}) \equiv [Ext]$, the induction hypothesis has to be applied twice, as the eigenvariable of the inference might occur in p or q. $\Box$

Lemma 2.3.2 *To each derivation $\mathfrak{D} \vdash t = s$, each variable x and each term p, we can effectively associate a derivation $\mathfrak{D}^* \vdash t[x/p] = s[x/p]$ which is a left (right, τ-free) derivation provided $\mathfrak{D}$ is such. Moreover, $\mathtt{s}(\mathfrak{D}^*) \leq \mathtt{s}(\mathfrak{D})$ and $\mathtt{h}(\mathfrak{D}^*) \leq \mathtt{h}(\mathfrak{D}) + \|p\|$.*

Proof Apply Lemma 2.3.1, after observing that for every term p we can easily construct a derivation $\mathfrak{D}_p \vdash^- p = p$ such that $\mathtt{s}(\mathfrak{D}_p) = 0$ and $\mathtt{h}(\mathfrak{D}_p) = \|p\|$. $\Box$

Remark 2.3.3 In view of the case $p \in V$ of the above Lemma, there is *no limitation* (w.r.t. to size, height, orientation and τ-freeness) in assuming, given a derivation $\mathfrak{D}$ and an $[Ext]$-inference R occurring in it, that the *eigenvariable* of R is distinct from those in an arbitrarily chosen finite set of variables. Henceforth, we will sometimes make a tacit use of this fact.

We conclude this section by showing how τ-free derivations can always be assumed, without limitations w.r.t. to *size* and *orientation*, to be in a certain *normal* form. This fact will be used to shorten some of the proofs in sect. 3.

To start with, notice the following immediate consequence of allowing *side terms* in the rules $[\mathsf{F}_l]$ and $[\mathsf{F}_r]$:

Fact 2.3.4 *If the conclusion $t = s$ can be obtained from the premise $t' = s \quad (t = s')$ by means of a left (right) combinatory rule then, for every q and for every finite list $\overline{p}$ of terms, the conclusion $t\overline{p} = q \quad (q = s\overline{p})$ can be obtained from the premise $t'\overline{p} = q$ $(q = s'\overline{p})$ by means of the same rule.*

This, in turn, implies that combinatory inferences whose conclusion is the left premise of an $[App]$ inference *can be permuted downwards*. Exploiting the latter fact leads to the following definition and result.

Definition 2.3.5 A derivation $\mathfrak{D}$ is *normal* provided it is τ-free, and is such that the left premise of each $[App]$ inference occurring in $\mathfrak{D}$ is not the conclusion of a combinatory inference.

Lemma 2.3.6 *To each τ-free derivation $\mathfrak{D} \vdash^- t = s$ we can effectively associate, preserving orientation, a normal derivation $\mathfrak{D}^* \vdash^- t = s$ with $\mathtt{s}(\mathfrak{D}^*) \leq \mathtt{s}(\mathfrak{D})$.*

Proof By induction on $\mathtt{h}(\mathfrak{D})$, taking cases according to $\mathsf{end}(\mathfrak{D})$. If the latter is *App*, then

$$\mathfrak{D} \quad \equiv \quad \dfrac{\mathfrak{D}_1\left\{\begin{array}{c}\vdots\\ t_1 = s_1\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ t_2 = s_2\end{array}\right\}\mathfrak{D}_2}{t_1 t_2 = s_1 s_2}\,{\scriptstyle App}\ .$$

Applying the I.H., we normalize $\mathfrak{D}_1$ and $\mathfrak{D}_2$ to $\mathfrak{D}_1^*$, resp. $\mathfrak{D}_2^*$. Next, let $t_1' = s_1'$ be the lowermost node in $\mathfrak{D}_1^*$ which is not the conclusion of a combinatory inference, and let $\mathfrak{D}_1'$ be the subderivation of $\mathfrak{D}_1^*$ ending with $t_1' = s_1'$. This means that the (possibly empty) path of $\mathfrak{D}_1^*$ going from $t_1' = s_1'$ to $t_1 = s_1$ consists only of combinatory inferences, say $R_1, \ldots R_n$. Then, in view of Fact 2.3.4, the following is a normal derivation of $t_1 t_2 = s_1 s_2$ satisfying $\mathtt{s}(\mathfrak{D}^*) \leq \mathtt{s}(\mathfrak{D})$:

$$\mathfrak{D}^* \quad = \quad \begin{array}{c}\dfrac{\mathfrak{D}_1'\left\{\begin{array}{c}\vdots\\ t_1' = s_1'\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ t_2 = s_2\end{array}\right\}\mathfrak{D}_2^*}{t_1' t_2 = s_1' s_2}\,{\scriptstyle App}\\ \vdots{\scriptstyle R_1 \ldots R_n}\\ t_1 t_2 = s_1 s_2\end{array}$$

The remaining cases are straightforward. □

3 Transitivity elimination for $\mathbf{G_e}[\mathbb{X}]$ proof systems

Throughout this section, we refer to an arbitrarily fixed combinatory system $\mathbb{X}$ and to the associated analytic *extensional* proof-system $\mathbf{G_e}[\mathbb{X}]$.

3.1 Overview of the proof

Transitivity elimination for analytic combinatory proof systems $\mathbf{G}[\mathbb{X}]$ *without extensionality* is demonstrated in [15] by following the (perhaps most natural) strategy which consists in showing how a *topmost* application of $[\tau]$ can be eliminated from a derivation:

$$\mathfrak{D}_1 \vdash^- t = s\,,\ \mathfrak{D}_2 \vdash^- s = r \quad \longmapsto \quad \mathfrak{D}^* \vdash^- t = r\,. \tag{$\Diamond$}$$

To prove ($\Diamond$), we argue by ω^3-induction[6], taking cases according to the possible combinations $\langle \mathsf{end}(\mathfrak{D}_1), \mathsf{end}(\mathfrak{D}_2)\rangle$ of the final inferences of $\mathfrak{D}_1$ and $\mathfrak{D}_2$.

Such a strategy can be suitably modified to cope with transitivity elimination for *extensional* systems $\mathbf{G_e}[\mathbb{X}]$, *provided* **X** *is linear* ([15], sect. 5). But it doesn't work any more when the linearity restriction is dropped — nor does it work with analytic proof systems for $\lambda\beta$- and $\lambda\beta\eta$-calculus, see [17]. Indeed, all kinds of inductive arguments we could reasonably imagine fail, when one considers the case of ($\Diamond$) in which

[6] Namely, by main induction on $\mathtt{h}'(\mathfrak{D}_1) + \mathtt{h}'(\mathfrak{D}_2)$, secondary induction on $\mathtt{s}(\mathfrak{D}_1) + \mathtt{s}(\mathfrak{D}_2)$, and ternary induction on $\|s\|$, where $\mathtt{h}'(\mathfrak{D})$ differs from $\mathtt{h}(\mathfrak{D})$ by not taking into account the applications of [*App*].

– $\mathsf{end}(\mathfrak{D}_1)$ is an [*App*] inference whose left premise is either the conclusion of an [*Ext*] inference, or the conclusion of another [*App*] inference whose left premise is the conclusion of an [*Ext*] inference, or ...
– $\mathsf{end}(\mathfrak{D}_2)$ is a left F-introduction rule and F is *non linear*,

as well as the symmetric case (see also the explanation in [17], sect. 3).

Luckily enough, a way out is found by taking as a starting point and then exploiting the more elaborate strategy we used in [17] to prove transitivity elimination for analytic proof systems for $\lambda\beta$- and $\lambda\beta\eta$-calculus. Essentially, the trick is to break down the symmetry between $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in $(\Diamond)$, by proving the following stronger *Main Lemma* (Lemma 3.4.1 below):

$$\mathfrak{D}_1 \vdash^- t = s\,,\ \mathfrak{D}_2 \vdash^- \Phi[\![s]\!] = r \quad \longmapsto \quad \mathfrak{D}^* \vdash^- \Phi[\![t]\!] = r\,, \qquad (\blacklozenge)$$

where Φ is an *arbitrary* context in $\mathbf{H_X}$. Clearly, $(\Diamond)$ (and in turn transitivity elimination) follows by taking $\Phi \equiv *$.

Yet the demonstration of $(\blacklozenge)$ is far from simple, and is actually much more delicate in the case of combinatory logic with extensionality than in the case of $\lambda\beta\eta$-calculus.

We shall argue by main induction on $\mathtt{s}(\mathfrak{D}_1)$, taking cases according to $R :=$ $\mathsf{end}(\mathfrak{D}_1)$. If the latter is a structural rule, or [*Ext*], additional subinductions on $\|s\|$ and $\mathtt{h}(\mathfrak{D}_2)$ are required, together with a clever handling of contexts and, when $\mathsf{end}(\mathfrak{D}_1)$ is [*Ext*], a careful distinction of many subcases. If instead R is a combinatory rule, here is what exactly is needed:

- the τ*-free* admissibility of a rule of *generalized (left)* F*-inversion:*

$$\mathfrak{D} \vdash^- \Phi[\![\mathsf{F}s_1 \dots s_{k_\mathsf{F}}\overline{p}]\!] = r \quad \longmapsto \quad \mathfrak{D}^* \vdash^- \Phi[\![d_\mathsf{F}[s_1,\dots,s_{k_\mathsf{F}}]\overline{p}]\!] = r\,, \qquad (\blacktriangle)$$

- and the τ*-free* admissibility of a rule of *generalized (left)* F*-introduction:*

$$\mathfrak{D} \vdash^- \Phi[\![d_\mathsf{F}[s_1,\dots,s_{k_\mathsf{F}}]\overline{p}]\!] = r \quad \longmapsto \quad \mathfrak{D}^* \vdash^- \Phi[\![\mathsf{F}s_1 \dots s_{k_\mathsf{F}}\overline{p}]\!] = r\,. \qquad (\blacktriangledown)$$

Also the proof of $(\blacktriangle)$ and, to a lesser extent, that of $(\blacktriangledown)$, require non trivial efforts, as we will see in a moment. But the game is worth the candle. For, coming back to $(\blacklozenge)$, if $\mathsf{end}(\mathfrak{D}_1)$ is $[\mathsf{F}_r]$ we may conclude by applying the M.I.H. to the subderivation of the premise of $[\mathsf{F}_r]$ and the derivation resulting from $\mathfrak{D}_2$ by *generalized* F*-inversion*. And, if $\mathsf{end}(\mathfrak{D}_1)$ is $[\mathsf{F}_l]$, we can first apply the M.I.H. to the subderivation of the premise of $[\mathsf{F}_r]$ and the derivation $\mathfrak{D}_2$, and then conclude by applying the rule of *generalized* F*-introduction*.

Summing up, the entire proof of transitivity elimination articulates in the following three main steps, in this order (which is essential). In § 3.2 we prove $(\blacktriangle)$, by making use of a kind of *marking* (or *indexing*) technique (Lemma 3.2.2). In § 3.3, by using $(\blacktriangle)$, we prove a weaker form of $(\Diamond)$ in which $\mathfrak{D}_1$ is assumed to be a *left* derivation (Lemma 3.3.1), and then use the latter to prove $(\blacktriangledown)$. Finally, in § 3.4 we prove the Main Lemma $(\blacklozenge)$ along the lines indicated above.

3.2 First step: τ-free admissibility of generalized inversion rules

Let F be a primitive combinator in $\mathbf{X}$, having index k and definition d_{F} under $\mathbb{X}$. Intuitively, by an F-*marked term* we mean nothing but a term t in which a number (possibly zero) of (possibly nested) occurrences of redexes of the form $\mathsf{F}t_1\ldots t_k$ have been marked in some way, so that it makes sense to speak of the (unmarked) term $t^\sharp$ obtained from t by replacing each marked redex occurrence $\mathsf{F}t_1\ldots t_k$ by the corresponding contractum $d_{\mathsf{F}}[t_1,\ldots,t_k]$. Formally, however, it is convenient to treat F-marked terms as expressions belonging to the set $\mathbf{T}^{\mathsf{F}}_{\mathbf{X}} \supset \mathbf{T}_{\mathbf{X}}$ which is defined by the grammar

$$\mathbf{T}^{\mathsf{F}}_{\mathbf{X}} = \mathbf{X}\,|\,\mathbf{V}\,|\,(\mathbf{T}^{\mathsf{F}}_{\mathbf{X}}\mathbf{T}^{\mathsf{F}}_{\mathbf{X}})\,|\,\lceil \mathsf{F}\,\overbrace{\mathbf{T}^{\mathsf{F}}_{\mathbf{X}}\ldots\mathbf{T}^{\mathsf{F}}_{\mathbf{X}}}^{k\ times}\rceil\,.$$

Capital letters P,Q,T (possibly with sub- and superscripts) range over F-marked terms, while $\overline{P},\overline{Q}$ range over finite lists of F-marked terms. Note that whereas every $t \in \mathbf{T}_{\mathbf{X}}$ is either an atom or an application term, F-marked terms partition into atoms (primitive combinators and variables), application terms PQ and *marked* F-*redexes* $\lceil \mathsf{F}T_1\ldots T_k\rceil$.

Given a F-marked term T, we denote by $|T|$ the $\mathbf{T}_{\mathbf{X}}$-term which is obtained from T by replacing every occurrence of the markers '$\lceil$' and '$\rceil$' with '(' and ')', respectively. If $|T| \equiv t$, we also say that T is *a* F-*marking of* t.

Finally, for every $T \in \mathbf{T}^{\mathsf{F}}_{\mathbf{X}}$, the term $T^\sharp \in \mathbf{T}_{\mathbf{X}}$ is defined inductively as follows:

- $x^\sharp := x$ and $\mathsf{G}^\sharp := \mathsf{G}$, for $x \in V$ and $\mathsf{G} \in \mathbf{X}$,
- $(PQ)^\sharp := P^\sharp Q^\sharp$,
- $\lceil \mathsf{F}P_1\ldots P_k\rceil^\sharp := d_{\mathsf{F}}[P_1^\sharp \ldots P_k^\sharp]$.

Lemma 3.2.1 *For every $T,P \in \mathbf{T}^{\mathsf{F}}_{\mathbf{X}}$ and every $x \in \mathbf{V}$:*

(i) $V(T^\sharp) \subseteq V(T) = V(|T|)$; (ii) $(T[x/P])^\sharp \equiv T^\sharp[x/P^\sharp]$.

Proof By straightforward induction on the construction of T. □

Lemma 3.2.2 *Let F be a primitive combinator having index k. Given a τ-free derivation $\mathfrak{D} \vdash^- t = s$ and a F-marking T of t, we can find a τ-free derivation*

$$\mathfrak{D}^* \vdash^- T^\sharp = s$$

which, additionally, is a right derivation provided $\mathfrak{D}$ is a right derivation.

Proof By main induction on $\mathsf{s}(\mathfrak{D})$ and secondary induction on $\|t\|$. In view of Lemma 2.3.6, we can assume without limitations that $\mathfrak{D}$ is *normal*. Lemma 3.2.1 will also be used, often without explicit mention.

We distinguish cases according to $R = \mathsf{end}(\mathfrak{D})$.

► Case A: R is $[\rho']$.

Then either T is an atom (so $T \equiv |T| \equiv s$), and the conclusion is trivial, or $k = 0$ (i.e. F has index 0) and T is $\lceil \mathsf{F} \rceil$. Then $t \equiv s \equiv \mathsf{F}$ and $T^\sharp \equiv d_\mathsf{F}$. We take

$$\mathfrak{D}^* := \dfrac{\begin{matrix} \vdots \scriptstyle\text{ structural rules} \\ d_\mathsf{F} = d_\mathsf{F} \end{matrix}}{d_\mathsf{F} = \mathsf{F}}\ \mathsf{F}_r\ .$$

Note that *left-handedness is not preserved in this case*.

► Case B: R is a *right* combinatory rule. Immediate by the M.I.H.[7]

► Case C: R is $[\mathsf{G}_l]$, with $\mathsf{G} \in \mathbf{X}$.
Let m be the index of G. Then there are $\mathbf{T_X}$-terms $t_1, \ldots, t_m, \overline{q}$ and corresponding F-markings $T_1, \ldots, T_m, \overline{Q}$ such that

$$t \equiv \mathsf{G}t_1 \ldots t_m \overline{q}\,, \quad \textit{and} \quad \mathfrak{D} \equiv \dfrac{\left.\begin{matrix} \vdots \\ d_G[t_1, \ldots, t_m]\overline{q} = s \end{matrix}\right\}\mathfrak{D}_1}{\mathsf{G}t_1 \ldots t_m \overline{q} = s}\ \mathsf{G}_l\ .$$

Also,

- *either* $T \equiv \mathsf{G}T_1 \ldots T_m \overline{Q}$, and so $T^\sharp \equiv \mathsf{G}T_1^\sharp \ldots T_m^\sharp \overline{Q}^\sharp$,
- *or* G is F, $m = k$ *and* $T \equiv \lceil \mathsf{G}T_1 \ldots T_k \rceil \overline{Q}$, so $T^\sharp \equiv d_\mathsf{G}[T_1^\sharp, \ldots, T_m^\sharp]\overline{Q}^\sharp$.

In the first case, by the M.I.H. applied to the subderivation $\mathfrak{D}_1$ w.r. to the F-marking $d_\mathsf{G}[T_1, \ldots, T_m]\overline{Q}$ of $d_G[t_1, \ldots, t_m]\overline{q}$, we get a derivation

$$\mathfrak{D}_1^* \vdash^- d_\mathsf{G}[T_1^\sharp, \ldots, T_m^\sharp]\overline{Q}^\sharp = s\,,$$

whence a final $[\mathsf{G}_l]$ inference yields the desired derivation $\mathfrak{D}^* \vdash^- \mathsf{G}T_1^\sharp \ldots T_m^\sharp \overline{Q}^\sharp = s$.
In the second case we proceed as above, but take $\mathfrak{D}^* := \mathfrak{D}_1^*$.

► Case D: R is $[Ext]$. Then:

$$\mathfrak{D} \equiv \dfrac{\left.\begin{matrix} \vdots \\ tx = sx \end{matrix}\right\}\mathfrak{D}_1}{t = s}\ Ext \qquad \textit{where } x \notin V(ts)\,.$$

From $|T| \equiv t$ we get $|Tx| \equiv tx$. Also, $(Tx)^\sharp \equiv T^\sharp x$, and $x \notin V(T^\sharp)$ by Lemma 3.2.1.
By the M.I.H. applied to the subderivation $\mathfrak{D}_1$ w.r. to the F-marking Tx of tx, we obtain a derivation $\mathfrak{D}_1^* \vdash^- T^\sharp x = sx$ and in turn, by a final application of $[Ext]$, the conclusion.

► Case E: R is $[App]$.
Then t, but possibly not T, is an application term. Hence we distinguish:

- Subcase E.1: there are $\mathbf{T_X}$-terms t_1, t_2 and corresponding F-markings T_1, T_2 such that

[7] Main induction hypothesis. S.I.H. = secondary induction hypothesis.

$$t \equiv t_1 t_2\,, \quad T \equiv T_1 T_2\,, \quad T^\sharp \equiv T_1^\sharp T_2^\sharp\,, \quad \textit{and} \quad \mathfrak{D} \equiv \cfrac{\mathfrak{D}_1 \left\{ \begin{matrix} \vdots \\ t_1 = s_1 \end{matrix} \right. \qquad \left. \begin{matrix} \vdots \\ t_2 = s_2 \end{matrix} \right\} \mathfrak{D}_2}{t_1 t_2 = s_1 s_2}\,{\scriptstyle App}\,.$$

The conclusion easily follows by applying the S.I.H. to $\mathfrak{D}_1$ and $\mathfrak{D}_2$.

- Subcase E.2: there are $\mathbf{T_X}$-terms $t_1, \dots, t_k$ and corresponding F-markings $T_1, \dots, T_k$ s.t.:

- $k \geq 1$, $t \equiv \mathsf{F}t_1 \dots t_k$, $T \equiv \lceil \mathsf{F}T_1 \dots T_k \rceil$, $T^\sharp \equiv d_{\mathsf{F}}[T_1^\sharp, \dots, T_k^\sharp]$,
- and for some i $(0 \leq i < k)$:

$$\mathfrak{D} \equiv \cfrac{\cfrac{\cfrac{\mathfrak{D}_0 \left\{ \cfrac{\vdots}{\mathsf{F}t_1 \dots t_i = r}\,{\scriptstyle R'} \right. \qquad \left. \begin{matrix} \vdots \\ t_{i+1} = r_{i+1} \end{matrix} \right\} \mathfrak{D}_{i+1}}{\mathsf{F}t_1 \dots t_{i+1} = r r_{i+1}}\,{\scriptstyle App}}{\begin{matrix} \ddots \\ \mathsf{F}t_1 \dots t_{k-1} = r r_{i+1} \dots r_{k-1} \end{matrix}} \qquad \left. \begin{matrix} \vdots \\ t_k = r_k \end{matrix} \right\} \mathfrak{D}_k}{\mathsf{F}t_1 \dots t_k = r r_{i+1} \dots r_k}\,{\scriptstyle App}$$

where $s \equiv r r_{i+1} \dots r_k$, and $\mathsf{F}t_1 \dots t_i = r$ is the lowermost node in the leftmost branch of $\mathfrak{D}$ which is the conclusion of an inference R' different from $[App]$ (note that this node possibly coincides with the left premise of the final inference of $\mathfrak{D}$, precisely when $i = k - 1$).

Now, $\mathfrak{D}$ is (assumed to be) *normal*. Therefore, the inference R' is either $[\rho']$ or $[Ext]$.

◇ E.2.1: R' is $[\rho']$. Necessarily $i = 0$, $r \equiv \mathsf{F}$, and so $s \equiv \mathsf{F}r_1 \dots r_k$.

Consider then the derivations $\mathfrak{D}_i \vdash^- t_i = r_i$, $1 \leq i \leq k$: we have $\mathtt{s}(\mathfrak{D}_i) \leq \mathtt{s}(\mathfrak{D})$, but $\|t_i\| < \|t\|$ as $t \equiv \mathsf{F}t_1 \dots t_k$. So, by applying to the $\mathfrak{D}_i$'s the S.I.H. w.r.t. to the F-marking T_i of t_i we get derivations $\mathfrak{D}_i^* \vdash^- T_i^\sharp = r_i$ for $1 \leq i \leq k$ and in turn, by making use of Lemma 2.3.1, a derivation

$$\mathfrak{D}^+ \vdash^- d_{\mathsf{F}}[T_1^\sharp, \dots, T_k^\sharp] = d_{\mathsf{F}}[r_1, \dots, r_k]\,.$$

A final application of $[\mathsf{F}_r]$ yields a τ-free derivation $\mathfrak{D}^*$ of $d_{\mathsf{F}}[T_1^\sharp, \dots, T_k^\sharp] = \mathsf{F}r_1 \dots r_k$, i.e. of $T^\sharp = s$, as requested. *Note that left-handedness is not preserved in this case.*

◇ E.2.2: R' is $[Ext]$. Then the subderivation $\mathfrak{D}_0$ looks like

$$\cfrac{\left. \begin{matrix} \vdots \\ \mathsf{F}t_1 \dots t_i x = r x \end{matrix} \right\} \mathfrak{D}_0'}{\mathsf{F}t_1 \dots t_i = r}\,{\scriptstyle Ext}$$

where the *eigenvariable* x doesn't occur in $t_1, \dots, t_i, r$ and (by Remark 2.3.3) can be assumed not to occur also in $t_{i+1}, \dots t_k, r_{i+1}, \dots, r_k$.

If $i < k-1$, let

$$\mathfrak{E} := \cfrac{\cfrac{\mathfrak{D}'_0\left\{\begin{array}{c}\vdots\\ \mathsf{F}t_1\ldots t_i x = rx\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ t_{i+2} = r_{i+2}\end{array}\right\}\mathfrak{D}_{i+2}}{\mathsf{F}t_1\ldots t_i x t_{i+2} = rxr_{i+2}}\,App \qquad \ddots}{\cfrac{\mathsf{F}t_1\ldots t_i x t_{i+2}\ldots t_{k-1} = rxr_{i+2}\ldots r_{k-1} \qquad \left.\begin{array}{c}\vdots\\ t_k = r_k\end{array}\right\}\mathfrak{D}_k}{\mathsf{F}t_1\ldots t_i x t_{i+2}\ldots t_k = rxr_{i+2}\ldots r_k}\,App}\,.$$

Otherwise ($i = k-1$) let $\mathfrak{E} := \mathfrak{D}'_0$.
Since an application of $[Ext]$ has been removed, we have $\mathtt{s}(\mathfrak{E}) < \mathtt{s}(\mathfrak{D})$. So we may apply to $\mathfrak{E}$ the M.I.H. with respect to the F-marking $\lceil \mathsf{F}T_1\ldots T_i x T_{i+2}\ldots T_k\rceil$ of $\mathsf{F}t_1\ldots t_i x t_{i+2}\ldots t_k$, to get a derivation

$$\mathfrak{E}^* \vdash^- d_{\mathsf{F}}[T_1^\sharp,\ldots,T_i^\sharp,x,T_{i+2}^\sharp,\ldots,T_k^\sharp] = rxr_{i+2}\ldots r_k\,.$$

On the other side, since $\|t_{i+1}\| < \|t\|$, we can apply the S.I.H. to $\mathfrak{D}_{i+1}$ w.r. to the marking T_{i+1} of t_{i+1}, to get a derivation

$$\mathfrak{D}^*_{i+1} \vdash^- T^\sharp_{i+1} = r_{i+1}.$$

Finally, by *parallel substitution* (Lemma 2.3.1) applied to $\mathfrak{E}^*$ and $\mathfrak{D}^*_{i+1}$ we get the desired derivation $\mathfrak{D}^* \vdash^- d_{\mathsf{F}}[T_1^\sharp,\ldots,T_k^\sharp] = rr_{i+1}\ldots r_k$.

It is easy to check that, in all cases, the transformation $\mathfrak{D} \mapsto \mathfrak{D}^*$ preserves *right-handedness*.

The key result of this section is now an immediate consequence of the above Lemma.

Lemma 3.2.3 (τ-free admissibility of generalized F-inversion rules) *For every $\mathsf{F} \in \mathbf{X}$ and every context $\Phi \in \mathbf{H}_{\mathbf{X}}$, the following inference rules are admissible in $\mathbf{G}^-_{\mathbf{e}}[\mathbb{X}]$:*

$$\frac{\Phi[\![\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\overline{p}]\!] = s}{\Phi[\![d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]\overline{p}]\!] = s}\,\mathsf{F}_l^{\mathrm{inv}}\,, \qquad \frac{s = \Phi[\![\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\overline{p}]\!]}{s = \Phi[\![d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]\overline{p}]\!]}\,\mathsf{F}_r^{\mathrm{inv}}\,.$$

Moreover, $[\mathsf{F}_l^{\mathrm{inv}}]$ preserves right-handedness and $[\mathsf{F}_r^{\mathrm{inv}}]$ preserves left-handedness.

Proof Given a τ-free (right) derivation $\mathfrak{D} \vdash^- \Phi[\![\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\overline{p}]\!] = s$ of the premise of $[\mathsf{F}_l^{\mathrm{inv}}]$, we obtain a τ-free (right) derivation $\mathfrak{D}^*$ of the conclusion $\Phi[\![d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]\overline{p}]\!] = s$ of $[\mathsf{F}_l^{\mathrm{inv}}]$ by applying Lemma 3.2.2 to $\mathfrak{D}$ w.r. to the F-marking

$$\Phi[\![\lceil \mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\rceil\overline{p}]\!] \quad \text{of the term} \quad \Phi[\![\mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\overline{p}]\!]\,,$$

since by Lemma 3.2.1

$$(\Phi[\![\lceil \mathsf{F}t_1\ldots t_{k_{\mathsf{F}}}\rceil\overline{p}]\!])^\sharp \equiv \Phi[\![d_{\mathsf{F}}[t_1,\ldots,t_{k_{\mathsf{F}}}]\overline{p}]\!]\,.$$

In turn, the admissibility of $[\mathsf{F}_r^{\mathrm{inv}}]$ follows from that of $[\mathsf{F}_l^{\mathrm{inv}}]$ by making use of the transformation $\mathfrak{D} \mapsto \widetilde{\mathfrak{D}}$. □

3.3 Second step: τ-free admissibility of generalized introduction rules

Lemma 3.3.1 (Left elimination) *To any given pair*

$$\mathfrak{D}_1 \vdash_L^- t = s \qquad \textit{and} \qquad \mathfrak{D}_2 \vdash^- s = r$$

of τ-free derivations, such that $\mathfrak{D}_1$ is a left derivation, *we can effectively associate a τ-free derivation*

$$\mathfrak{D}^* \vdash^- t = r$$

which, moreover, is a left *derivation provided that also $\mathfrak{D}_2$ is a* left *derivation.*

Proof By main induction on $\mathtt{s}(\mathfrak{D}_2)$, secondary induction on $\mathtt{s}(\mathfrak{D}_1)$ and ternary induction on the depth of the 'cut-term', $\|s\|$.

Let $R_i := \mathsf{end}(\mathfrak{D}_i)$, $i = 1,2$. As $\mathfrak{D}_1$ is, by assumption, a *left* derivation, it is easily seen that the following is an exhaustive list of the cases to be considered:

- Case A: either R_1 or R_2 is $[\rho']$;
- Case B: R_1 is $[\mathsf{G}_l]$ or R_2 is $[\mathsf{G}_r]$, for some $\mathsf{G} \in \mathbf{X}$;
- Case C: R_1 is $[App]$ and R_2 is $[App]$;
- Case D: either R_1 or R_2 is $[Ext]$;
- Case E: R_2 is $[\mathsf{G}_l]$, for some $\mathsf{G} \in \mathbf{X}$.

Now, Case A is trivial, while B is easily dealt with by the M.I.H. (in case R_2 is $[\mathsf{G}_r]$) or the S.I.H. (in case R_1 is $[\mathsf{G}_l]$). As for C, apply the ternary I.H. two times. Let us discuss in detail the remaining two cases.

► Case D.
Suppose R_1 is $[Ext]$. Then

$$\mathfrak{D}_1 \equiv \quad \cfrac{\left.\begin{array}{c}\vdots\\ tx = sx\end{array}\right\}\mathfrak{D}_1^1}{t = s}\,{\scriptstyle Ext}\,,$$

where x doesn't occur in t,s and (by Remark 2.3.3) can also be assumed not to occur in r. Consider the derivation

$$\mathfrak{D}_2' \equiv \quad \cfrac{\mathfrak{D}_2\left\{\begin{array}{c}\vdots\\ s = r\end{array}\right. \qquad x = x}{sx = rx}\,{\scriptstyle App}\,.$$

We have $\mathtt{s}(\mathfrak{D}_2') = \mathtt{s}(\mathfrak{D}_2)$, while $\mathtt{s}(\mathfrak{D}_1^1) < \mathtt{s}(\mathfrak{D}_1)$. So we can apply the S.I.H. to $\mathfrak{D}_1^1$ and $\mathfrak{D}_2'$, to get a τ-free derivation

$$\mathfrak{E} \vdash^- tx = rx\,.$$

A final application of $[Ext]$ yields the desired $\mathfrak{D}^* \vdash^- t = r$.
If R_2 is $[Ext]$ we reason symmetrically (of course, in this case the M.I.H. is applied).

► Case E.
Then, for some $\mathsf{G} \in \mathbf{X}$ with $m = m_{\mathsf{G}}$, and some terms $s_1 \dots s_m, \overline{p}$, we have $s \equiv \mathsf{G} s_1 \dots s_m \overline{p}$,

$$\mathfrak{D}_1 \equiv \quad \begin{array}{c} \vdots \\ t = \mathsf{G} s_1 \dots s_m \overline{p} \end{array} \qquad \textit{and} \qquad \mathfrak{D}_2 \equiv \quad \dfrac{\left.\begin{array}{c} \vdots \\ d_{\mathsf{G}}[s_1, \dots, s_m]\overline{p} = r \end{array}\right\} \mathfrak{D}_2'}{\mathsf{G} s_1 \dots s_m \overline{p} = r}\ \mathsf{G}_l\,.$$

By applying to the conclusion of $\mathfrak{D}_1$ (w.r. to the context $*$) the admissible rule $[\mathsf{F}_r^{\mathrm{inv}}]$ of Lemma 3.2.3, we get a τ-free *left* derivation

$$\mathfrak{D}_1' \vdash_L^- t = d_{\mathsf{G}}[s_1, \dots, s_m]\overline{p}\,.$$

Notice that possibly $\mathtt{s}(\mathfrak{D}_1') > \mathtt{s}(\mathfrak{D}_1)$! However, since $\mathtt{s}(\mathfrak{D}_2') < \mathtt{s}(\mathfrak{D}_2)$, we can apply the M.I.H. to $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$ to get a derivation $\mathfrak{D}^* \vdash^- t = r$.
By inspecting the whole proof it is easily verified that $\mathfrak{D}^*$ is, in all cases, a *left* derivation provided $\mathfrak{D}_2$ is such. □

We can now prove:

Lemma 3.3.2 (τ-free admissibility of generalized F-introduction rules) *For every* $\mathsf{F} \in \mathbf{X}$ *and every context* $\Phi \in \mathbf{H}_{\mathbf{X}}$*, the following inference rules are admissible in* $\mathbf{G}_{\mathbf{e}}^-[\mathbb{X}]$*:*

$$\frac{\Phi[\![d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p}]\!] = s}{\Phi[\![\mathsf{F} t_1 \dots t_{k_{\mathsf{F}}} \overline{p}]\!] = s}\ \mathsf{F}_l^+\,, \qquad\qquad \frac{s = \Phi[\![d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p}]\!]}{s = \Phi[\![\mathsf{F} t_1 \dots t_{k_{\mathsf{F}}} \overline{p}]\!]}\ \mathsf{F}_r^+\,.$$

Proof Suppose we are given a τ-free derivation $\mathfrak{D} \vdash^- \Phi[\![d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p}]\!] = s$.
By applying Lemma 2.3.1 to the τ-free *left* derivations

$$\mathfrak{E}_1 := \quad \begin{array}{c} \vdots\ \text{structural rules} \\ \Phi[\![x]\!] = \Phi[\![x]\!] \end{array} \qquad \text{and} \qquad \mathfrak{E}_2 := \quad \dfrac{\begin{array}{c} \vdots\ \text{structural rules} \\ d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p} = d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p} \end{array}}{\mathsf{F} t_1 \dots t_{k_{\mathsf{F}}} \overline{p} = d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p}}\ \mathsf{F}_l\,,$$

where $x \notin V(\Phi)$, we get a τ-free *left* derivation

$$\mathfrak{E} \vdash_L^- \Phi[\![\mathsf{F} t_1 \dots t_{k_{\mathsf{F}}} \overline{p}]\!] = \Phi[\![d_{\mathsf{F}}[t_1, \dots, t_{k_{\mathsf{F}}}]\overline{p}]\!]\,.$$

Thus, by Lemma 3.3.1 applied to $\mathfrak{E}$ and $\mathfrak{D}$, we get a τ-free derivation

$$\mathfrak{D}^* \vdash^- \Phi[\![\mathsf{F} t_1 \dots t_{k_{\mathsf{F}}} \overline{p}]\!] = s\,.$$

This proves the claim about $[\mathsf{F}_l^+]$ and in turn, by making use of the transformation $\mathfrak{D} \mapsto \widetilde{\mathfrak{D}}$, the claim about $[\mathsf{F}_r^+]$. □

Although this fact will not be needed later on, notice that $[\mathsf{F}_l^+]$ preserves left-handedness and $[\mathsf{F}_r^+]$ preserves right-handedness.

3.4 Final step: main elimination lemma

We are now in a position to prove:

Lemma 3.4.1 (Main elimination lemma) *Given two τ-free derivations*

$$\mathfrak{D}_1 \vdash^- t = s \quad \textit{and} \quad \mathfrak{D}_2 \vdash^- \Phi[\![s]\!] = r$$

we can find a τ-free derivation

$$\mathfrak{D}^* \vdash^- \Phi[\![t]\!] = r\,.$$

Proof By main induction on $\mathtt{s}(\mathfrak{D}_1)$, secondary induction on $\|s\|$ and ternary induction on $\mathtt{h}(\mathfrak{D}_2)$, taking cases according to $R_1 := \mathsf{end}(\mathfrak{D}_1)$.

- ► Case A: R_1 is $[\rho']$. Then $t \equiv s$, and we take $\mathfrak{D}^* := \mathfrak{D}_2$.
- ► Case B: R_1 is $[App]$. Then $t \equiv t_1 t_2,\ s \equiv s_1 s_2$, and

$$\mathfrak{D}_1 \equiv \dfrac{\mathfrak{D}_1'\left\{\begin{array}{c}\vdots\\ t_1 = s_1\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ t_2 = s_2\end{array}\right\}\mathfrak{D}_1''}{t_1 t_2 = s_1 s_2}\,{\scriptstyle App}\,.$$

Let $\Psi := \Phi[\![(s_1 *)]\!]$, so that $\Psi[\![s_2]\!] \equiv \Phi[\![s]\!]$ and $\mathfrak{D}_2 \vdash^- \Psi[\![s_2]\!] = r$. Since $\mathtt{s}(\mathfrak{D}_1'') \leq \mathtt{s}(\mathfrak{D}_1)$ and $\|s_2\| < \|s\|$, we can apply the S.I.H. to $\mathfrak{D}_1''$ and $\mathfrak{D}_2$, giving a derivation:

$$\mathfrak{D}_3 \vdash^- \Psi[\![t_2]\!] = r\,, \qquad \textit{i.e.}\ \mathfrak{D}_3 \vdash^- \Phi[\![s_1 t_2]\!] = r\,.$$

Now, let $\Theta := \Phi[\![(* t_2)]\!]$, so that $\Theta[\![s_1]\!] \equiv \Phi[\![s_1 t_2]\!]$ and $\mathfrak{D}_3 \vdash^- \Theta[\![s_1]\!] = r$. Since $\mathtt{s}(\mathfrak{D}_1') \leq \mathtt{s}(\mathfrak{D}_1)$ and $\|s_1\| < \|s\|$, we can apply again the S.I.H., this time to $\mathfrak{D}_1'$ and $\mathfrak{D}_3$, and get

$$\mathfrak{D}^* \vdash^- \Theta[\![t_1]\!] = r\,, \qquad \textit{i.e.}\ \mathfrak{D}^* \vdash^- \Phi[\![t]\!] = r\,.$$

- ► Case C: R_1 is $[\mathsf{F}_l]$, for some $\mathsf{F} \in \mathbf{X}$. Then $t \equiv \mathsf{F} t_1 \ldots t_{k_\mathsf{F}} \overline{p}$, and $\mathfrak{D}_1$ has the form

$$\dfrac{\mathfrak{D}_1'\left\{\begin{array}{c}\vdots\\ d_\mathsf{F}[t_1, \ldots, t_{k_\mathsf{F}}]\overline{p} = s\end{array}\right.}{\mathsf{F} t_1 \ldots t_{k_\mathsf{F}} \overline{p} = s}\,{\scriptstyle \mathsf{F}_l}\,, \quad \textit{where}\ \mathtt{s}(\mathfrak{D}_1') < \mathtt{s}(\mathfrak{D}_1).$$

Using the $\mathbf{G}_\mathbf{e}^-[\mathbb{X}]$-admissible *generalized introduction* rule $[\mathsf{F}_r^+]$ of Lemma 3.3.2, we set:

$$\mathfrak{D}^* := \dfrac{\dfrac{\mathfrak{D}_1'\left\{\begin{array}{c}\vdots\\ d_\mathsf{F}[t_1, \ldots, t_{k_\mathsf{F}}]\overline{p} = s\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ \Phi[\![s]\!] = r\end{array}\right\}\mathfrak{D}_2}{\Phi[\![d_\mathsf{F}[t_1, \ldots, t_{k_\mathsf{F}}]\overline{p}]\!] = r}\,{\scriptstyle M.I.H.}}{\Phi[\![\mathsf{F} t_1 \ldots t_{k_\mathsf{F}} \overline{p}]\!] = r}\,{\scriptstyle \mathsf{F}_l^+}$$

▶ Case D: R_1 is $[\mathsf{F}_r]$, for some $\mathsf{F} \in \mathbf{X}$. Then $s \equiv \mathsf{F}s_1 \ldots s_{k_\mathsf{F}}\overline{q}$, and $\mathfrak{D}_1$ has the form

$$\dfrac{\mathfrak{D}'_1\left\{\begin{array}{c}\vdots\\ t = d_\mathsf{F}[s_1,\ldots,s_{k_\mathsf{F}}]\overline{q}\end{array}\right.}{t = \mathsf{F}s_1 \ldots s_{k_\mathsf{F}}\overline{q}}\,\mathsf{F}_l\,, \quad \textit{where } \mathtt{s}(\mathfrak{D}'_1) < \mathtt{s}(\mathfrak{D}_1).$$

Using the $\mathbf{G}^-_\mathbf{e}[\mathbb{X}]$-admissible *generalized inversion* rule $[\mathsf{F}^{\mathrm{inv}}_l]$ of Lemma 3.2.3, we set:

$$\mathfrak{D}^* := \dfrac{\mathfrak{D}'_1\left\{\begin{array}{c}\vdots\\ t = d_\mathsf{F}[s_1,\ldots,s_{k_\mathsf{F}}]\overline{q}\end{array}\right. \qquad \dfrac{\left.\begin{array}{c}\vdots\\ \Phi[\![\mathsf{F}s_1 \ldots s_{k_\mathsf{F}}\overline{q}]\!] = r\end{array}\right\}\mathfrak{D}_2}{\Phi[\![d_\mathsf{F}[s_1,\ldots,s_{k_\mathsf{F}}]\overline{q}]\!] = r}\,\mathsf{F}^{\mathrm{inv}}_l}{\Phi[\![t]\!] = r}\,M.I.H.$$

▶ Case E: R_1 is $[Ext]$. Then

$$\mathfrak{D}_1 \equiv \dfrac{\left.\begin{array}{c}\vdots\\ tx = sx\end{array}\right\}\mathfrak{D}'_1}{t = s}\,Ext \quad \textit{where } x \notin V(ts) \textit{ and } \mathtt{s}(\mathfrak{D}'_1) < \mathtt{s}(\mathfrak{D}_1).$$

Without limitations, $x \notin V(r)$ as well as $x \notin V(\Phi)$.
We have now to look at $R_2 := \mathsf{end}(\mathfrak{D}_2)$, and distinguish the corresponding subcases.

• Subcase E.1: R_2 is $[\rho']$.

Excluding the trivial case of Φ having no holes, we necessarily have that $\Phi \equiv *$, and $s \equiv r \in \mathbf{V} \cup \mathbf{X}$. We set:

$$\mathfrak{D}^* := \dfrac{\dfrac{\mathfrak{D}'_1\left\{\begin{array}{c}\vdots\\ tx = sx\end{array}\right. \qquad \dfrac{\mathfrak{D}_2\left\{\begin{array}{c}\vdots\\ s = r\end{array}\right. \qquad \dfrac{}{x = x}\,\rho'}{sx = rx}\,App}{tx = rx}\,M.I.H.}{t = r}\,Ext\,.$$

• Subcase E.2: R_2 is $[App]$.

If Φ has no holes or $\Phi \equiv *$, we proceed as in subcase E.1 above.
Otherwise, there are contexts Ψ_1, Ψ_2 s.t. $\Phi \equiv \Psi_1\Psi_2$, $\Phi[\![s]\!] \equiv \Psi_1[\![s]\!]\Psi_2[\![s]\!]$, $r \equiv r_1r_2$ for some r_1, r_2, and:

$$\mathfrak{D}_2 \equiv \dfrac{\mathfrak{D}'_2\left\{\begin{array}{c}\vdots\\ \Psi_1[\![s]\!] = r_1\end{array}\right. \qquad \left.\begin{array}{c}\vdots\\ \Psi_2[\![s]\!] = r_2\end{array}\right\}\mathfrak{D}''_2}{\Psi_1[\![s]\!]\Psi_2[\![s]\!] = r_1r_2}\,App\,, \qquad \textit{where } \mathtt{h}(\mathfrak{D}'_2)\,,\, \mathtt{h}(\mathfrak{D}''_2) < \mathtt{h}(\mathfrak{D}_2)\,.$$

Then, the ternary I.H. can be applied:

$$\mathfrak{D}^* := \dfrac{\dfrac{\mathfrak{D}_1\left\{\begin{array}{c}\vdots\\ t=s\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ \Psi_1[\![s]\!]=r_1\end{array}\right\}\mathfrak{D}'_2}{\Psi_1[\![t]\!]=r_1}\,{\scriptstyle T.I.H.} \qquad \dfrac{\mathfrak{D}_1\left\{\begin{array}{c}\vdots\\ t=s\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ \Psi_2[\![s]\!]=r_2\end{array}\right\}\mathfrak{D}''_2}{\Psi_2[\![t]\!]=r_2}\,{\scriptstyle T.I.H.}}{\Psi_1[\![t]\!]\Psi_2[\![t]\!]=r_1r_2}\,{\scriptstyle App}\,.$$

- Subcase E.3: R_2 is $[\mathsf{F}_r]$, for some $\mathsf{F}\in\mathbf{X}$.

We have $r\equiv \mathsf{F}r_1\ldots r_{k_\mathsf{F}}\overline{q}$ and

$$\mathfrak{D}_2 \equiv \quad \dfrac{\mathfrak{D}'_2\left\{\begin{array}{c}\vdots\\ \Phi[\![s]\!]=d_\mathsf{F}[r_1,\ldots,r_{k_\mathsf{F}}]\overline{q}\end{array}\right.}{\Phi[\![s]\!]=\mathsf{F}r_1\ldots r_{k_\mathsf{F}}\overline{q}}\,{\scriptstyle \mathsf{F}_r}$$

Again, the ternary I.H. can be applied to $\mathfrak{D}_1$ and $\mathfrak{D}'_2$:

$$\mathfrak{D}^* := \quad \dfrac{\dfrac{\mathfrak{D}_1\left\{\begin{array}{c}\vdots\\ t=s\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ \Phi[\![s]\!]=d_\mathsf{F}[r_1,\ldots,r_{k_\mathsf{F}}]\overline{q}\end{array}\right\}\mathfrak{D}'_2}{\Phi[\![t]\!]=d_\mathsf{F}[r_1,\ldots,r_{k_\mathsf{F}}]\overline{q}}\,{\scriptstyle T.I.H}}{\Phi[\![t]\!]=\mathsf{F}r_1\ldots r_{k_\mathsf{F}}\overline{q}}\,{\scriptstyle \mathsf{F}_r}$$

- Subcase E.4: R_2 is $[Ext]$.

Then

$$\mathfrak{D}_2 \equiv \quad \dfrac{\mathfrak{D}'_2\left\{\begin{array}{c}\vdots\\ (\Phi[\![s]\!])z=rz\end{array}\right.}{\Phi[\![s]\!]=r}\,{\scriptstyle Ext}\,, \quad \textit{where } z\notin V(\Phi[\![s]\!],r) \textit{ and } \mathtt{h}(\mathfrak{D}'_2)<\mathtt{h}(\mathfrak{D}_2)\,.$$

We apply the ternary I.H. to $\mathfrak{D}_1$ and $\mathfrak{D}'_2$ w.r. to the context $\Psi\equiv\Phi x$; that is (safely assuming $z\notin V(\Phi[\![t]\!])$), we set:

$$\mathfrak{D}^* := \quad \dfrac{\dfrac{\mathfrak{D}_1\left\{\begin{array}{c}\vdots\\ t=s\end{array}\right. \quad \left.\begin{array}{c}\vdots\\ (\Phi[\![s]\!])z=rz\end{array}\right\}\mathfrak{D}'_2}{(\Phi[\![t]\!])z=rz}\,{\scriptstyle T.I.H.}}{\Phi[\![t]\!]=r}\,{\scriptstyle Ext}$$

- Subcase E.5: R_2 is $[\mathsf{F}_l]$, for some $\mathsf{F}\in\mathbf{X}$.

Then, for some terms $q_1\ldots q_n$, $\Phi[\![s]\!]\equiv\mathsf{F}q_1\ldots q_n$, where $0\le k=k_\mathsf{F}\le n$; further,

$$\mathfrak{D}_2 \equiv \quad \dfrac{\mathfrak{D}'_2\left\{\begin{array}{c}\vdots\\ d_\mathsf{F}[q_1,\ldots,q_k]q_{k+1}\ldots q_n=r\end{array}\right.}{\mathsf{F}q_1\ldots q_n=r}\,{\scriptstyle \mathsf{F}_l}\,, \qquad \textit{where } \mathtt{h}(\mathfrak{D}'_2)<\mathtt{h}(\mathfrak{D}_2)\,.$$

Now, excluding the trivial case of Φ without holes, it easily follows from $\Phi[\![s]\!]\equiv\mathsf{F}q_1\ldots q_n$ that only the following three cases are possible as to the form of Φ:

◇ E.5.1: $\Phi \equiv *$ and $s \equiv \mathsf{F} q_1 \dots q_n$. We proceed exactly as in subcase E.1.

◇ E.5.2: $\Phi \equiv \mathsf{F} \Theta_1 \dots \Theta_n$ for some contexts $\Theta_1, \dots, \Theta_n$, and $q_i \equiv \Theta_i[\![s]\!]$ for $1 \leq 1 \leq n$. Then $\Phi[\![t]\!] \equiv \mathsf{F}(\Theta_1[\![t]\!]) \dots (\Theta_n[\![t]\!])$. Consider the context

$$\Psi := d_{\mathsf{F}}[\Theta_1, \dots, \Theta_k]\Theta_{k+1} \dots \Theta_n\,, \quad \textit{so that } \mathfrak{D}'_2 \vdash^- \Psi[\![s]\!] = r\,. \tag{3.1}$$

By the ternary I.H. applied to $\mathfrak{D}_1$ and $\mathfrak{D}'_2$ w.r. to the context Ψ we get a derivation

$$\mathfrak{D}_3 \vdash^- \Psi[\![t]\!] = r\,, \quad \textit{i.e. } \mathfrak{D}_3 \vdash^- d_{\mathsf{F}}[(\Theta_1[\![t]\!]), \dots, (\Theta_k[\![t]\!])](\Theta_{k+1}[\![t]\!]) \dots (\Theta_n[\![t]\!])\,,$$

whence a final application of $[\mathsf{F}_l]$ yields the desired $\mathfrak{D}^* \vdash^- \mathsf{F}(\Theta_1[\![t]\!]) \dots (\Theta_n[\![t]\!]) = r$. And now the most complicated of the three cases:

◇ E.5.3: $\Phi \equiv * \,\Theta_{m+1} \dots \Theta_n$ for some $\Theta_{m+1}, \dots, \Theta_n$ $(0 \leq m < n)$, *and* $s \equiv \mathsf{F} q_1 \dots q_m$. For $1 \leq i \leq m$, let $\Theta_i := q_i$, so that, *vacuously*,

$$\Theta_i[\![s]\!] \equiv \Theta_i[\![t]\!] \equiv q_i \quad (1 \leq i \leq m). \tag{3.2}$$

Let also the context Ψ be defined as in (3.1) above. Then

$$\mathsf{F} q_1 \dots q_n \equiv \Phi[\![s]\!] \equiv s(\Theta_{m+1}[\![s]\!]) \dots (\Theta_n[\![s]\!]) \equiv \mathsf{F} q_1 \dots q_m(\Theta_{m+1}[\![s]\!]) \dots (\Theta_n[\![s]\!])\,.$$

So also, in view of (3.2):

$$q_1 \equiv \Theta_1[\![s]\!]\,, \; \dots \;, q_n \equiv \Theta_n[\![s]\!]\,, \quad \text{and} \quad \mathfrak{D}'_2 \vdash^- \Psi[\![s]\!] = r \quad (\text{cf. } (3.1))\,,$$

while

$$\Phi[\![t]\!] \equiv t(\Theta_{m+1}[\![t]\!]) \dots (\Theta_n[\![t]\!])\,. \tag{3.3}$$

Thus we proceed as follows. First of all, exactly as in E.5.2, we apply the ternary I.H. to $\mathfrak{D}_1$ and $\mathfrak{D}'_2$ w.r. to Ψ, followed by a final application of $[\mathsf{F}_l]$, and get a derivation (now called $\mathfrak{E}$):

$$\mathfrak{E} \vdash^- \mathsf{F}(\Theta_1[\![t]\!]) \dots (\Theta_n[\![t]\!]) = r\,, \; \textit{i.e., by (3.2)}, \; \mathfrak{E} \vdash^- s(\Theta_{m+1}[\![t]\!]) \dots (\Theta_n[\![t]\!]) = r\,. \tag{3.4}$$

Thus, let $\Phi' := *(\Theta_{m+2}[\![t]\!]) \dots (\Theta_n[\![t]\!])$ (observe that possibly $\Phi' \equiv *$). By (3.4):

$$\mathfrak{E} \vdash^- \Phi'[\![s(\Theta_{m+1}[\![t]\!])]\!] = r\,. \tag{3.5}$$

On the other side, by applying Lemma 2.3.2 to $\mathfrak{D}'_1 \vdash^- tx = sx$ and the term $\Theta_{m+1}[\![t]\!]$ (which exists, because $m < n$!) we get a derivation

$$\mathfrak{D}''_1 \vdash^- t(\Theta_{m+1}[\![t]\!]) = s(\Theta_{m+1}[\![t]\!])\,, \qquad \textit{with } \mathtt{s}(\mathfrak{D}''_1) \leq \mathtt{s}(\mathfrak{D}'_1) < \mathtt{s}(\mathfrak{D}_1)\,.$$

A final application of the M.I.H. to $\mathfrak{D}''_1$ and $\mathfrak{E}$ (cf. (3.5)) w.r. to the context Φ' thus yields the desired derivation

$$\mathfrak{D}^* \vdash^- \Phi'[\![t(\Theta_{m+1}[\![t]\!])]\!] = r\,, \quad \textit{i.e., by 3.3}, \quad \mathfrak{D}^* \vdash^- \Phi[\![t]\!] = r\,.$$

□

Theorem 3.4.2 **(τ-elimination for $\mathbf{G_e}[\mathbb{X}]$)** *Every* $\mathbf{G_e}[\mathbb{X}]$*-derivation*

$$\mathfrak{D} \vdash t = s$$

can (effectively) be transformed into a τ-free $\mathbf{G_e}[\mathbb{X}]$*-derivation*

$$\mathfrak{D}^* \vdash^- t = s\,.$$

Proof By induction on the number of occurrences of rule $[\tau]$ in $\mathfrak{D}$, using the *Main Elimination Lemma* 3.4.1 with $\Phi \equiv *$ to eliminate a topmost occurrence of $[\tau]$. □

Remark 3.4.3 The proof system $\mathbf{G}_\xi[\mathbb{C}]$ is obtained from $\mathbf{G_e}[\mathbb{C}]$ by replacing the extensionality rule $[Ext]$ with the rule $[\xi]$, see sect. 1. Clearly, these two systems are equivalent. Now, as a consequence of the above Theorem 3.4.2 together with Fact 6.1 of [15], we may conclude — *indirectly* — that *also* $\mathbf{G}_\xi[\mathbb{C}]$ *admits transitivity elimination*.

4 Solution of Curry and Hindley's problem

Let us finally come to our solution of Curry and Hindley's problem, asking for a confluence proof of *combinatory strong reduction* $\succ$ which is independent of λ-calculus.

Henceforth, terms are applicative combinations of variables and the three basic combinators $\mathsf{I}, \mathsf{K}, \mathsf{S}$. Strong reduction $\succ$ between terms has already been defined in sect. 1. We just recall that, for every term t and variable x, the term $\lambda^* x.t$ is inductively defined as follows:

$$\lambda^* x.t := \begin{cases} \mathsf{I}, & \text{if } t \equiv x, \\ t', & \text{if } t \equiv t'x \text{ with } x \notin V(t'), \\ \mathsf{K}t, & \text{if } x \notin V(t); \\ \mathsf{S}(\lambda^* x.t_1)(\lambda^* x.t_2), & \text{if } t \equiv t_1 t_2 \text{ and the two previous cases do not apply.} \end{cases}$$

$\mathbf{C_e}[\mathbb{C}]$ ($\equiv \mathbf{CL_{ext}}$) is the standard synthetic proof system for full extensional combinatory logic, and $\mathbf{G_e}[\mathbb{C}]$ is the corresponding analytic proof system — see Example 2.2.4. As usual, we write $t =_{c\beta\eta} s$ short for $\mathbf{C_e}[\mathbb{C}] \vdash t = s$.

Lemma 4.1 **(Common $\succ$-reduct extraction)** *From any given τ-free* $\mathbf{G_e}[\mathbb{C}]$*-derivation*

$$\mathfrak{D} \vdash^- t = s$$

we can effectively extract a term $r_{\mathfrak{D}}$ such that

$$t \succ r_{\mathfrak{D}} \prec s\,.$$

Proof By straightforward induction on $\mathtt{h}(\mathfrak{D})$, taking cases according to $R := \mathsf{end}(\mathfrak{D})$. If R is an instance of $[\rho']$, say $t = t$, we take $r_{\mathfrak{D}} := t$. If R is a combinatory rule and $\mathfrak{D}_1$ is the subderivation of the premise, we can take $r_{\mathfrak{D}} := r_{\mathfrak{D}_1}$, as

$$\mathsf{S}p_1 p_2 p_3 \overline{q} \succ d_{\mathsf{S}}[p_1, p_2, p_3]\overline{q} \equiv p_1 p_3 (p_2 p_3)\overline{q}\,, \quad \text{and similarly for } \mathsf{K} \text{ and } \mathsf{I}.$$

If R is $[App]$ and $\mathfrak{D}_1, \mathfrak{D}_2$ are the subderivations of the left and right premise, we can clearly take $r_{\mathfrak{D}} := r_{\mathfrak{D}_1} r_{\mathfrak{D}_2}$. Finally, if R is $[Ext]$,

$$\mathfrak{D} \equiv \quad \dfrac{\mathfrak{D}_1 \left\{ \begin{array}{c} \vdots \\ tx = sx \end{array} \right.}{t = s}\, Ext \qquad with\ x \notin V(ts),$$

and by the I.H. $tx \succ r_{\mathfrak{D}_1} \prec sx$. Then, by the ξ-rule for $\succ$,

$$t \equiv \lambda^* x.tx \succ \lambda^* x.r_{\mathfrak{D}_1} \prec \lambda^* x.sx \equiv s\,,$$

and we can take $r_{\mathfrak{D}} := \lambda^* x.r_{\mathfrak{D}_1}$. □

Now, the promised solution:

Theorem 4.2 (Church-Rosser property for strong reduction — *direct* proof)

$$t =_{c\beta\eta} s \quad \Rightarrow \quad \exists r\, (t \succ r \prec s).$$

Proof Assume $t =_{c\beta\eta} s$. Then, by Proposition 2.2.6, there is a $\mathbf{G_e}[\mathbb{C}]$-derivation $\mathfrak{D} \vdash t = s$. Hence, by *transitivity elimination* (Theorem 3.4.2), there is a τ-free $\mathbf{G_e}[\mathbb{C}]$-derivation

$$\mathfrak{D}^* \vdash^- t = s\,.$$

The conclusion $t \succ r_{\mathfrak{D}^*} \prec s$ now follows by applying the *extraction* Lemma 4.1. □

It is evident, by simple inspection of the above proof as well as of the proofs of all the results therein employed, that *no* detour *has been made through λ-calculus*. Also, it goes without saying that, *mutatis mutandis*, the same proof does work in case I is not taken as a primitive combinator, but defined as $\mathsf{I} := \mathsf{SKK}$.

Final remark 4.3 Just for the sake of completeness, we would like to conclude with a sketch of Curry's *indirect* proof of $CR(\succ)$. Let

$$(\ )_\lambda : \mathbf{T}_{\{\mathsf{I},\mathsf{K},\mathsf{S}\}} \longrightarrow \Lambda \qquad \text{and} \qquad (\ )_H : \Lambda \longrightarrow \mathbf{T}_{\{\mathsf{I},\mathsf{K},\mathsf{S}\}}$$

be the standard translations of combinatory terms into λ-terms, and conversely (see e.g. [9], sect. 9 B). These satisfy, as it is easily verified, the three properties:

(P1) for $t \in \mathbf{T}_{\{\mathsf{I},\mathsf{K},\mathsf{S}\}}$: $\quad (t_\lambda)_H \equiv t\,,$
(P2) for $t, s \in \Lambda$: $\quad t \twoheadrightarrow_{\beta\eta} s \;\Rightarrow\; t_H \succ s_H\,,$
(P3) for $t, s \in \mathbf{T}_{\{\mathsf{I},\mathsf{K},\mathsf{S}\}}$: $\quad t =_{c\beta\eta} s \;\Rightarrow\; t_\lambda =_{\beta\eta} s_\lambda\,,$

where $\twoheadrightarrow_{\beta\eta}$ and $=_{\beta\eta}$ are $\lambda\beta\eta$-reduction, resp. $\lambda\beta\eta$-convertibility.

Assuming $t =_{c\beta\eta} s$, we have $t_\lambda =_{\beta\eta} s_\lambda$ by (P3), so *by the Church-Rosser Theorem for $\lambda\beta\eta$-reduction* there is a $r \in \Lambda$ such that $t_\lambda \twoheadrightarrow_{\beta\eta} r$ and $s_\lambda \twoheadrightarrow_{\beta\eta} r$. By using (P2) and (P1), we conclude $t \succ r_H \prec s$.

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